THE PROCEEDINGS

OF THE

UNITED STATES NAVAL INSTITUTE.

Vol. X. No. 1. 1884. Whole No. 28.

NAVAL INSTITUTE, ANNAPOLIS, MD.

JANUARY, 1884.

RESEARCHES ON THE EFFECTS OF POWDER.

(1874–1878.)

BY M. E. SARRAU, Ingénieur des Poudres et Salpêtres.

TRANSLATED BY LIEUTENANTS J. F. MEIGS AND R. R. INGERSOLL, U. S. N.

PART I.

THEORETICAL RESEARCHES ON THE EFFECTS OF GUNPOWDER AND OTHER EXPLOSIVES.

CHAPTER I.

PROPERTIES OF GASES.

1. *Mariotte's and Gay-Lussac's Laws.*—Gases and superheated vapors tend towards a limiting state, called that of a perfect gas, which is characterized by the two following laws:

1. *Mariotte's Law.*—The pressures of the same mass of gas are inversely proportional to the volumes.

2. *Gay-Lussac's Law.*—All gases have the same coefficient of dilatation under constant pressure, and this coefficient is independent of the pressure.

These two laws are expressed by the equation,

$$pv = K(1 + \alpha t); \tag{1}$$

where v is the volume of unit of weight of the gas, p the pressure, t the temperature, α the coefficient of dilatation under constant pressure, and K a constant depending upon the nature of the gas.

2. *Absolute Temperature.*—The coefficient of dilatation a, which is the same for all perfect gases according to Gay-Lussac's law, is $\frac{1}{273}$, nearly, according to the experiments of Regnault. Consequently, equation (1) becomes,

$$pv = \frac{K}{273}(273 + t), \tag{2}$$

The factor $(273 + t)$ is called the absolute temperature in the dynamic theory of heat. It is the centigrade temperature counted from a zero placed 273 degrees below the ordinary zero.

3. *Specific Volume.*—If we call p_0 the normal atmospheric pressure, and put $K = p_0 v_0$, equation (2) may be written,

$$pv = \frac{p_0 v_0}{273}(273 + t), \tag{3}$$

and, in this form, we see that v_0 is the value of v which corresponds to the values $t = 0$ and $p = p_0$ of the temperature and pressure.

This constant, which is called the specific volume, represents then the volume of unit of weight of the gas at zero temperature, and under the normal pressure; if the gas under consideration can reach these conditions of pressure and temperature without change of state and without ceasing to satisfy Mariotte's and Gay-Lussac's laws, its numerical value is the reciprocal of the specific weight, or weight of the unit of volume at zero and under the pressure p_0.

The specific volume ceases evidently to have this signification for gases and vapors which are in the state of a perfect gas only at temperatures above zero. Its definition results from equation (3), which the gas under consideration must satisfy in the limiting state; its numerical value may be derived from that equation by determining by experiment, in the physical conditions where the equation is applicable, a system of values corresponding to p, v, and t.

The determination of specific volumes, or, what is the same thing, the densities of gases and vapors, is of great importance in chemistry; and they have been the object of the research of many eminent scientists. We shall return, at the end of this chapter, to the physical laws which have been established, dwelling particularly upon those which are useful in the approximate calculation of the force of explosive substances.

4. *Specific Heat.*—The specific heat of a substance is the quantity of heat necessary to raise unit of weight of the substance one degree centigrade. The quantity may be measured in two ways: the body which is being heated may be allowed to expand freely under a deter-

mined pressure, or the volume of the body may be maintained constant; in the first case the specific heat is of constant pressure, and in the second of constant volume.

Calorimetric experiments have established the following law, which, like those of Mariotté and Gay-Lussac, appears to characterize the state of the perfect gas:

The specific heats under constant pressure and constant volume are independent of the pressure and volume.

This law has been experimentally verified by Regnault by the direct determination of specific heats under constant pressure.

As to the specific heats for constant volume, their direct measurement being almost impossible, this verification cannot be made. The law appears, however, sufficiently confirmed by the indirect measurement which has been made, of the relation between the specific heats throughout a considerable range of temperature and pressure, by many investigators.

5. *Conversion of Heat into Work by the Expansion of Gas.*—The above laws being established, we may deduce as follows the relation that exists between the variations of volume and pressure of a gas and the heat necessary to produce them.

Resuming the fundamental equation (3), and writing for shortness,

$$R = \frac{p_0 v_0}{273}, \tag{4}$$

and calling T the absolute temperature $273 + t$, this equation may be written,

$$pv = RT. \tag{5}$$

The consequences which follow are these:

1st. If, the pressure p remaining constant, the volume varies by the amount dv, the temperature undergoes a corresponding variation represented by $\frac{pdv}{R}$; and, consequently, the gas has received a quantity of heat $\frac{c'pdv}{R}$, c' being the specific heat under constant pressure.

2d. If, the volume v remaining constant, the pressure varies by dp, the temperature varies by $\frac{vdp}{R}$; and the gas has received a quantity of heat equal to $\frac{cvdp}{R}$, c being the specific heat under constant volume.

3d. Consequently, if the volume and pressure increase together by dv and dp, the gas receives a quantity of heat:

$$dq = \frac{1}{R}(c'pdv + cvdp). \tag{6}$$

Differentiating (5), we have,

$$RdT = pdv + vdp, \tag{7}$$

and, eliminating successively between (6) and (7) dp and dv, we have the two equations,

$$dq = cdT + \frac{c' - c}{R} pdv, \tag{8}$$

$$dq = c'dT - \frac{c' - c}{R} vdp; \tag{9}$$

which, together with (6), contain all the thermodynamic laws of gases.

6. We will first consider equation (8). The quantity pdv, which appears in its second member, is evidently the work done by the elastic force of the gas when the volume increases by dv.

Let ω be an infinitely small element of the surface which incloses the gas. The pressure on this element is $p\omega$, since p is the pressure on unit of surface; and the element of work of this pressure corresponding to an infinitely small increase of volume is $p\omega h$, h being the displacement of ω perpendicular to itself. The work done then is $p\Sigma\omega h$; but $\Sigma\omega h$ is the variation of the volume dv; therefore pdv is the external work done by the gas.

It results, then, from equation (8) that the quantity of heat absorbed by a gas in an infinitely small change is composed of two terms; of which the first is proportional to the change of temperature, and the second to the element of external work.

If we consider a change which alters by finite quantities the temperature and volume of a gas, we derive, by integrating (8), the quantity of heat absorbed during the change:

$$q = \int \left(cdT + \frac{c' - c}{R} pdv \right).$$

If c and c' are functions of T and v, this expression can only be integrated if we know the relation between them; but if we assume (No. 4) that c and c' are constants, we have

$$q = c \int dT + \frac{c' - c}{R} \int pdv,$$

or calling T_0 and T_1 the initial and final temperatures and $\mathcal{E}$ the total external work done by the elastic force of the gas, we have

$$q = c(T_1 - T_0) + \frac{c' - c}{R} \mathcal{E}. \tag{10}$$

6½. *Equivalence of the heat and work.*—If we put $T_1 = T_0$ in the last equation, that is, if the gas returns to its initial state, we have

$$q = \frac{c' - c}{R} \mathcal{E}. \tag{11}$$

Thus, in this case, the amount of heat absorbed by the gas is proportional to the external work done.

The quantity

$$A = \frac{c' - c}{R}, \tag{12}$$

which expresses the ratio between the heat absorbed and the work done, is called the calorific equivalent of the work; and its reciprocal,

$$E = \frac{R}{c' - c}, \tag{13}$$

is the mechanical equivalent of the heat.

We thus see that we may deduce from the known laws which govern gases the notion of the equivalence of heat and work, whose precise conception has led to so great progress in the theory of heat. The fundamental postulate of this theory consists in the assertion that the quantity which has been designated E is invariable, and *independent of the nature of the body* which has served as intermediate in the transformation of heat into work. We need not recall here the reasoning which has established the soundness of this principle, which may otherwise be considered as sufficiently confirmed by the verification of its numerous consequences.

We consider, then, that, for perfect gases, there exists between the volume and the two specific heats the relation expressed by (13); or, taking into consideration the value of R (4), the relation,

$$E = \frac{1}{273} \frac{p_0 v_0}{c' - c} \tag{14}$$

expresses an invariable number, depending solely upon the choice of the units which serve to express the quantities of work and heat.

7. *Value of the mechanical equivalent of heat.*—Since the value of E is independent of the nature of the gas, we may derive it from the values of v_0, c', and c, which have been experimentally determined for many gases, following the laws of Mariotte and Gay-Lussac. We take, for example, atmospheric air.

Regnault's experiments show that under constant pressure the specific heat of air is $c' = .23754$,

and that its specific volume (the reciprocal of the specific weight), taking for units the meter and the kilogram, is

$$v_0 = \frac{1}{1.2932}.$$

Finally, the ratio between the two specific heats, deduced from the velocity of sound as observed by Regnault, is

$$\frac{c'}{c} = 1.3945,$$

from which we derive, for the specific heat of constant volume,

$$c = .17034.$$

Inserting these numerical values in (14) and putting $p_0 = 10333$, we find

$$E = 436. \tag{15}$$

This, then, is the value of the mechanical equivalent of heat, as determined from the best determined constants which are now obtainable. It is the value which we shall adopt. We must not lose sight of the fact, however, that there is some uncertainty as to its exact value, in consequence of the large result that small variations in the value of the ratio $\frac{c'}{c}$ would produce.

8. *Adiabatic transformations.*—Up to this point we have considered only those transformations in the state of a gas which are caused by its losing a quantity of heat. It may, however, happen that the transformation takes place within an envelope which is impermeable to heat; in which case the gas may change its volume and pressure, and consequently its temperature, without gain or loss of heat. The change is then said to be adiabatic, and the laws which govern it may be derived from equations (6), (8), and (9), by putting $dq = 0$. We thus obtain the relations,

$$c' p dv + c v dp = 0, \tag{16}$$

$$c dT + \frac{c' - c}{R} p dv = 0, \tag{17}$$

$$c' dT - \frac{c' - c}{R} v dp = 0; \tag{18}$$

from which follow several important results.

9. *Law of pressures.*—Taking equation (16), and putting $n = \frac{c'}{c}$, it may be written

$$n \frac{dv}{v} + \frac{dp}{p} = 0,$$

whence, integrating, and calling f a constant,

$$n \log v + \log p = \log f,$$

or,

$$v^n p = f. \tag{19}$$

Also v, the volume of the unit of weight, is the reciprocal of ρ, the weight of the unit of volume. Thus, equation (19) may be written $p = f \rho^n$; or, in an adiabatic change, the pressure varies proportionally to a power of the density equal to the ratio of the two specific heats.

Such is the law of pressures discovered by Laplace and Poisson before the birth of the mechanical theory of heat.

10. *Law of temperatures.*—Let us consider equation (17). If we put $c' = nc$, and for p its value $\frac{RT}{v}$ derived from (5), it becomes

$$\frac{dT}{T} + (n-1)\frac{dv}{v} = 0,$$

whence, by integration, and calling f_1 a constant,

(20) $$Tv^{n-1} = f_1.$$

Consequently, recollecting that, as before, v is the reciprocal of the density, we have this law: *The absolute temperature of a gas, in any adiabatic transformation, is proportional to a power of the density equal to the ratio of the two specific heats minus one.*

We may also present this law under another form. Equation (17) may be written $$Ec\,dT + p\,dv = 0,$$

by recollecting that we have, from (13), $\frac{c'-c}{R} = \frac{1}{E}$. Consequently, integrating, and calling T_0 and T_1 the initial and final temperatures of the gas we have

(21) $$Ec(T_0 - T_1) = \int p\,dv.$$

But the second member represents the total work done by the elastic force of the gas (No. 5). Thus, in an adiabatic transformation, *the temperature is lowered by a quantity which is proportional to the external work done.*

11. *The work done by the indefinite adiabatic expansion of gas.*—If we suppose that the gas expands indefinitely without gain or loss of heat, the volume v increases indefinitely; and it results, from equation (20), that the temperature T approaches zero. The final temperature of the transformation T_1 is then zero. Putting $T_1 = 0$ in equation (21), we have, for the total work of the expansion,

(22) $$\theta = Ec\,T_0.$$

The work is therefore equal to the product of the mechanical equivalent of heat, the specific heat under constant volume, and the initial absolute temperature.

We may thus say that the measure of this work is found by multiplying the mechanical equivalent of heat by the quantity of heat which the gas absorbs, under constant volume, when its temperature is raised from the absolute zero to T_0; or gives out, under constant volume, when its temperature is lowered from T_0 to absolute zero.

We shall finish this exposition by a rapid summary of the laws which control the volumes and specific heats of gases and vapors,

and which permit us, in many cases, to calculate these important elements without its becoming necessary to have recourse to experimentation.

12. *Laws governing mixtures of gases.*—1st. The specific volume of a mixture of gases is the compound mean of the specific volumes of the gases mixed. Thus, let a_1, a_2, a_3, be the quantities of the various gases which make up a unit of weight of the mixture, and v_1, v_2, v_3, the specific volumes corresponding; the specific volume of the mixture is

$$v_0 = a_1 v_1 + a_2 v_2 + a_3 v_3 + \quad (23)$$

EXAMPLE.—Taking for units the kilogram and meter, the specific volumes of oxygen and nitrogen are .69941 and .79607. And, according to the determinations of Dumas and Boussingault, these gases exist in atmospheric air in the ratio of .23 to .77. Substituting these values in (23), we have, for the specific volume of atmospheric air, $v_0 = .77385$. Regnault found by direct experiment, $v_0 = .77318$, which differs little from the first.

2d. The specific heat (under constant pressure or constant volume) of a mixture of gases is the compound mean of the specific heats of the gases mixed. This law gives rise to the formula,

$$c = a_1 c_1 + a_2 c_2 + a_3 c_3 + \quad (24)$$

similar to the foregoing.

If we apply this to the heat of atmospheric air under constant pressure, using the values $c'_1 = .2175$ and $c'_2 = .2438$ for the specific heats of oxygen and nitrogen, we find $c' = .2378$. Regnault's experimental determination was .2374.

13. *Relation between the specific volumes and molecular weights.*—Gay-Lussac was the first to formulate the following important law, which subsequent experimentation has fully confirmed:

The product of the specific volume of a gaseous body by its chemical equivalent, or by a multiple or simple submultiple of this equivalent, is constant.

Thus, if we designate by v_0 and e the specific volume and the equivalent, we have

$$mev_0 = h, \quad (25)$$

m being a number which is generally unity, sometimes 2 or $\frac{1}{2}$, and very rarely $\frac{1}{4}$.

Taking $m = 1$ for hydrogen, we have:

$m=1$ for nitrogen, chlorine, bromine, iodine, and all the metals which have as yet been volatilized. To this group belong also numerous binary compounds; protoxide of nitrogen, carbonic oxide, carbonic dioxide, vapor of water, sulphurous acid, hydrogen sulphide, &c.

$m=2$ for some simple bodies: Oxygen, sulphur, selenium, tellurium, phosphorus, arsenic.

$m=\frac{1}{2}$ for binoxide of nitrogen, chlorhydric acid, ammonia.

$m=\frac{1}{4}$ for some very few substances, chlorhydrate of ammonia, for example.

If we take unity as the equivalent of hydrogen, and take the kilogram and meter as units, the numerical value of h is 11.15, about; so that the specific volume of a gas whose equivalent is e is given by the formula,

$$v_0=\frac{11.15}{me}. \tag{26}$$

Gay-Lussac's law may be very simply presented, if we admit, following the atomic theory, that the number of molecules in unit of volume, at a fixed pressure and temperature, is the same for all gases. In fact, according to this hypothesis, the weight of unit of volume is proportional to the weight of a molecule, or to the molecular weight. But the weight of unit of volume, and the specific volume v_0, are inversely proportional; hence, from (26), the molecular weight is proportional to me.

Thus, the multiple or submultiple of the equivalent which satisfies the law of the specific volumes is a measure of the molecular weight. For example, the molecular weight of hydrogen and nitrogen being H and N, those of oxygen and sulphur are O^2 and S^2. The molecular weight of ammonia and chlorhydrate of ammonia, for which the values of m are $\frac{1}{2}$ and $\frac{1}{4}$, are $N^{\frac{1}{2}}H^{\frac{3}{2}}$ and $N^{\frac{1}{4}}HCl^{\frac{1}{4}}$.

We may thus formulate as follows the law of specific volumes: *The product of the specific volume of a gas by its molecular weight is constant.*

14. *Laws of specific heats.—Molecular heat.*—The product of the molecular weight of a body by its specific heat is called the molecular heat of the body.

Regarding gases we have from experiment:

1st. That the molecular heat under constant pressure has the same value, K', for hydrogen, nitrogen, and oxygen.

2d. That it is greater than K' for the other gases and vapors, and that it becomes greater as the gas becomes more complex.

The numerical value of K', derived by taking the mean of the specific heats of hydrogen and nitrogen, may be taken as 3.411.

If, then, we call c' the specific heat under constant pressure and ω the molecular weight, we may write,

$$c' = \frac{K' + a}{\omega}, \tag{27}$$

a being a quantity which is equal to zero for the simple gases, and those the nearest to the state of a perfect gas; and is positive for others, and increases as the molecule of the gas or vapor is made up of a larger number of atomic elements.

For example, for vapor of water, carbonic sulphide, alcohol, and ether, whose molecular weights are

$$HO \quad CS^2 \quad C^4H^6O \quad C^4H^5O,$$

the values of a are respectively

$$.913 \quad 2.555 \quad 7.017 \quad 14.386.$$

15. The specific heat under constant volume, whose experimental determination is almost impossible, is calculated for perfect gases by the relation (14), which gives

$$c = c' - \frac{1}{273}\frac{p_0 v_0}{E}. \tag{28}$$

It is only necessary to know c' and the specific volume v_0. The latter may be determined by experiment, or deduced from formula (26), when the molecular weight ω is known. We have, in this case, $v_0 = \frac{h}{\omega}$; and consequently putting

$$K = K' - \frac{1}{273}\frac{p_0 h}{E}, \tag{29}$$

and replacing c' by its value (27) in equation (28), the latter may be written,

$$c = \frac{K + a}{\omega}. \tag{30}$$

The value of K, the molecular heat under constant volume of simple perfect gases, is obtained from numerical values of quantities on which (29) depends.

We have already shown that

$K' = 3.411$ $\quad h = 11.15$ $\quad E = 436$ $\quad p_0 = 10333$,

whence it results $K = 2.443$.

16. *Relation between the two specific heats.*—This ratio, which is so important a factor in all adiabatic transformations, is deduced from (27) and (30); which give

$$n = \frac{c'}{c} = \frac{K' + a}{K + a}. \tag{31}$$

For hydrogen, nitrogen, and oxygen we have, very nearly, $\alpha = 0$; consequently

$$n = \frac{K'}{K} = \frac{3.411}{2.443} = 1.40, \text{ nearly.}$$

For gases and vapors whose molecules are complex, the value of n approaches rapidly to unity for increasing values of α.

If we admit, for example, the values of α already given (No. 14), we have, for vapor of water, sulphur, carbon, alcohol, and ether:

$$n = 1.29 \qquad 1.19 \qquad 1.10 \qquad 1.06.$$

17. Finally, the value $n = 1.40$, which we have found for those of the simple gases which approach most nearly the ideal state which has been called the state of a perfect gas, appears itself inferior to the theoretical limit which would be realized if the molecules of the gas were strictly mono-atomic. It results, in fact, from reasoning based on the new thermodynamic theory, that the ratio of the two specific heats has in this case the extreme value $n = \frac{5}{3}$. The smaller value (1.40) found is explained by Gehrard and Clausius by the hypothesis that the molecules of even these gases contain at least two atoms.

We are thus led to conclude that the ratio of the two specific heats, variable with the greater or less complexity of the body, is between the limits $\frac{5}{3}$ and 1; and approaches rapidly to the inferior limit when the number of atoms making up the molecule is increased. We shall see hereafter that these considerations, which may appear at first purely speculative, are not without importance for the particular object in view.

18. *Hypothetical law of Clausius.*—It is known from experiments of Regnault upon the specific heats of solids that:

1st. The molecular heat of simple bodies is constant (Dulong and Petit's law).

2d. The specific heat of compound bodies is the compound mean of that of the simple bodies of which they are composed.

Clausius supposes that these two laws may be applied to the specific heats under constant volume of perfect gases. According to this hypothesis the specific heat under constant volume of a perfect gas may be found by the relation,

$$c = \frac{K'}{\omega}, \tag{32}$$

ω being its molecular weight and K' the constant, whose definition and value have already been given (No. 15).

19. The specific heat of a compound gas is deduced, by formula (24), from its chemical composition, *as if the combined elements were mixed.* If, for example, a gas whose composition is represented by the formula

$$\omega_1^{n_1} \quad \omega_2^{n_2} \quad \omega_3^{n_3},$$

ω_1, ω_2, ω_3 being the molecular weights of the constituents, so that

$$\omega = n_1\omega_1 + n_2\omega_2 + n_3\omega_3$$

will be the molecular weight of the compound. We apply here formula (24), remarking that the quantities of the elements in unit of weight of the compound are

$$a_1 = \frac{n_1\omega_1}{\omega}, \qquad a_2 = \frac{n_2\omega_2}{\omega}, \qquad a_3 = \frac{n_3\omega_3}{\omega}.$$

Consequently, calling c_1, c_2, c_3, the specific heats under constant volume of the constituents, and c that of the compound, we have

$$c = \frac{n_1\omega_1c_1 + n_2\omega_2c_2 + n_3\omega_3c_3}{\omega},$$

and finally, from the relation (32),

$$c = \frac{K(n_1 + n_2 + n_3)}{\omega}. \tag{33}$$

Putting $n_1 + n_2 + n_3 = N$, and putting $v_0 = \frac{h}{\omega}$ and $c = \frac{KN}{\omega}$ in equation (28), and from (29) we have,

$$c' = \frac{K' + (N-1)K}{\omega}. \tag{34}$$

From (33) and (34) we have, finally,

$$n = \frac{c'}{c} = 1 + \frac{1}{N}\left(\frac{K'}{K} - 1\right); \tag{35}$$

which, from the value $\frac{K'}{K} = 1.40$, already found (No. 16), for perfect simple gases, may be written in the very simple form,

$$n = 1 + \frac{2}{5N}. \tag{36}$$

20. It is hardly necessary to dwell upon the importance of the preceding formulas. They show that the specific heats of gases and vapors depend upon their chemical composition.

The application of these formulas, even to perfect gases, must, however, be under certain reservations. For if, as shown by the experiments of Regnault, hydrogen, nitrogen, and oxygen satisfy sufficiently accurately (32), it is not so for chlorine and bromine; and the departures from the law are sufficiently great to make us attribute them to something else than alterations in the laws of Mariotte and Gay-Lussac.

Also, formula (33) gives greater values to the specific heats of compound gases than those found by experiment. In this last case, however, the error must be lessened at high temperatures; for the specific heats under constant pressure of imperfect gases increases with the temperature, as Regnault has shown by experiment upon carbonic acid.

If it is not, then, certain that the hypothesis of Clausius gives rigorous formulas in the limiting state, we can at least admit that it furnishes an approximation equally good with that furnished by the experimental laws which govern the specific heats of simple and compound solids; and that the relations which are derived from it, so frequently used by Berthelot in his researches in thermo-chemistry, may be reasonably accepted.

We shall apply them in the following chapter to the theoretical calculation of the principal elements which are required for the approximate evaluation of the force of explosive substances.

CHAPTER II.

On the Force of Explosive Substances.

1. When a substance is exploded in a capacity of constant volume, the products of combustion develop a pressure whose value, variable with the time, reaches a maximum in an extremely short space of time, and then decreases progressively in consequence of cooling. At the end of this cooling the substance is changed into products whose nature is variable with the conditions of the explosion, and which may be determined by chemical analysis.

These products are, according to circumstances:

Exclusively gaseous (detonation of chloride of nitrogen); gaseous and solid (explosives of the nitrate and chlorate class); gaseous and liquid (nitro-glycerine and gun-cotton).

In all cases we must admit that, at the instant of maximum tension, the products are totally gasified; the temperature at this instant is generally greater than the melting point of the most refractory metals.

Also, at this temperature, the products of combustion are probably not the same as those found after the cooling. As remarked long since by Melsens, they are then partially or totally dissociated; and, by a chain of transformations producing gradually less simple combi-

nations, the variable state of temperature and pressure brings about the final state, which alone can be the subject of observation.

2. *Force of an Explosive Substance.*—If we suppose that the laws of Mariotte and Gay-Lussac are applicable to the gasified products of explosion at the instant of maximum tension, the value of this tension, as a function of the temperature and specific volume corresponding, is easily obtained.

Suppose, for example, that we consider unit of weight of an explosive detonating in volume v. The maximum pressure will be given by the equation

$$pv = \frac{p_0 v_0 T_0}{273}, \tag{37}$$

in which T_0 is the absolute temperature of the products of explosion at the instant of maximum tension; v_0 the specific volume of the mixture of these products, supposed entirely gasified.

Consequently, we have the pressure, which is inversely proportional to the volume, equal to $\frac{p_0 v_0 T_0}{273}$, when the volume is reduced to unity.

The quantity

$$f = \frac{p_0 v_0 T_0}{273} \tag{38}$$

is what we shall call the force of the explosive. It represents *the pressure developed by the unit of weight of the substance detonating in unit of volume.*

We shall next see how its approximate value may be determined.

3. *Heat and Temperature of Combustion.*—We shall call heat of combustion the quantity of heat Q that unit of weight of the explosive substance evolves, under constant volume, in the ideal case where the final temperature of the products of combustion is absolute zero.

The absolute temperature of combustion is the temperature T, which the products would have if all the heat of the combustion were used to heat them from absolute zero.

If the specific heat of the products of combustion under constant volume were constant throughout the transformations which these products undergo in passing from absolute zero to T, or, inversely, from T to absolute zero, we would have between Q and T the well-known relation

$$T = \frac{Q}{c}. \tag{39}$$

The specific heat, c, would be practically constant if, in the changes undergone, the products remained gaseous, and if, in this state, the hypothesis of Clausius were really applicable to them. In fact the specific heat of a compound gas being, according to this hypothesis, the same as if the combined elements were mixed, it is clear that the mean specific heat of the mixture preserves, in all states, a value equal to that of the entirety of the simple bodies which make up the explosive body, supposed freely mixed.

In reality, there is reason to believe (Chapter I, No. 14) that this is only approximate, and that the specific heat increases in transformations in which the constituents pass from a less to a greater state of complexity. It also certainly increases at the time of the passage of a body from the gaseous to the liquid state, and from the liquid to the solid state.

Consequently, equation (39) would give too great a value for T, if we take for c a value derived by the hypothesis of Clausius for the entirety of the products supposed gaseous. We would have too small a value on the other hand if we follow Bunsen and Schisckhoff, and take the value relative to the final state.

We shall adopt the first value, which, as Berthelot has remarked, seems to be preferable for reactions whose temperatures are high, and which, moreover, has the advantage of enabling us to calculate the value *a priori* from the composition of the explosive substance and independently of an analysis of the products of explosion.

4. We shall suppose also, following all the authors who have treated subjects analogous to the one we have in view, that the temperature, T_0, corresponding to the maximum pressure, is sensibly equal to the temperature of combustion, T, already defined. We write, consequently,

$$T_0 = \frac{Q}{c}. \tag{40}$$

Thus we have a superior limit; for the heat of combustion, Q, comprises not only the quantities of heat which change the temperature of the products, but also those which the chemical or physical changes of state produce.

These principles being admitted, we obtain the definite expression of the force of an explosive substance in this form,

$$f = \frac{1}{273} \cdot \frac{p_0 v_0 Q}{c}. \tag{40bis}$$

The force, then, *is directly proportional to the heat of combustion and the specific volume, and inversely proportional to the specific heat.*

It remains now to show how these three characteristic elements may be theoretically or experimentally determined.

5. *Value of the specific heat under constant volume.*—According to the hypothesis which has been admitted, the value of c is obtained by applying the law of gaseous mixtures (Chapter I, No. 12) to the elements which make up the substance under consideration.

If, for example, a_1, a_2, a_3, are the proportions of the various elements, and ω_1, ω_2, ω_3, their molecular weights, for unit of weight, the corresponding specific heats are, according to equation (32),

$$\frac{K}{\omega_1}, \quad \frac{K}{\omega_2}, \quad \frac{K}{\omega_3},$$

and we have, consequently, according to formula (24), for the specific heat sought,

$$c = K\left(\frac{a_1}{\omega_1} + \frac{a_2}{\omega_2} + \frac{a_3}{\omega_3} + \dots\right) \tag{41}$$

This formula requires that the molecular weight of the elements shall be known. The following are the values for those simple bodies which make up the most common explosives:

Name of body.	Molecular weight. Symbol.	Value.
Hydrogen	H	1
Nitrogen	N	14
Chlorine	Cl	35·5
Oxygen	O^2	16
Sulphur	S^2	32
Carbon	C^2	12
Potassium	K^2	19·5
Sodium	Na^2	11·5

6. Another method may be given for the calculation of c. Suppose, first, that the substance is a definite compound, like nitroglycerine or gun-cotton; and that its equivalent e is represented by the formula $\omega_1^{n_1}\,\omega_2^{n_2}\,\omega_3^{n_3}$; ω_1, ω_2, ω_3, being the molecular weights of the elements, so that we have

$$e = n_1\omega_1 + n_2\omega_2 + n_3\omega_3 +$$

The proportions of the elements in unit of weight being

$$a_1 = \frac{n_1\omega_1}{e}, \quad a_2 = \frac{n_2\omega_2}{e}, \quad a_3 = \frac{n_3\omega_3}{e},$$

formula (41) becomes

$$c = \frac{K(n_1 + n_2 + n_3 + \dots)}{e}. \tag{42}$$

Notice that, in this expression, as well as in the equivalent one (41), the numerical value of K is (Chapter I, No. 15),

$$K = 2.443.$$

Let us take, for example, chloride of nitrogen, NCl^3; we have

$$e = 14 + 3 \times 35.5 = 120.5,\ n_1 = 1,\ n_2 = 3,\ c = \frac{2.443 \times 4}{120.5} = .0811.$$

Again, let nitro-glycerine be represented by the formula

$$C^6H^5(NO^6H)^3 = C^6H^5N^3O^{18};$$

in this case, $e = 227$.

Writing the formula in the form

$$(C^2)^3H^5N^3(O^2)^9,$$

in order to show the molecular weights, we have

$$c = \frac{2.443\,(3+5+3+9)}{227} = .2197.$$

7. When the substance is a mixture we may first calculate the specific heat for each of the constituents, and then take the compound mean according to the proportions of the ingredients.

The following table shows the values of c for the simple and compound bodies which compose the principal explosives:

Name of substance.	Equivalent.	Value of c.
Sulphur	S	.0763
Carbon	C	.2036
Nitrate of potassium	NO^5KO	.1451
Nitrate of sodium	NO^5NaO	.1724
Chlorate of potassium	ClO^5KO	.1197
Chloride of nitrogen	NCl^3	.0811
Nitro-glycerine	$C^6H^5(NO^6H)^3$	.2197
Gun-cotton	$C^{24}H^{10}O^{10}NO^6H^5$	.2314
Picrate of potassium	$C^{12}H^2K(NO^4)^3O^2$	.1830

The following are the values of c for the various powders made in France:

Kind of powder.	Mixture. Nitre.	Sulphur.	Charcoal.	Value of c.
Best sporting powder	78	10	12	.1452
Cannon powder	75	12.5	12.5	.1437
Fine-grained powder, called 1 B . .	74	10.5	15.5	.1468
Powder of commerce	72	13	15	.1448
Ordinary blasting powder	62	20	18	.1420

These values differ very little, and may all be taken as equivalent to their mean, .1445.

It has also been found that for a mixture of 55 parts of picrate of potassium and 45 of saltpetre, $c = .1661$; and, for a mixture of equal weights of picrate and chlorate of potassium, $c = .1513$.

NOTE.—In the calculations relative to gunpowder the charcoal has been treated as pure carbon. The charcoal used in the manufacture of gunpowder contains, however, hydrogen and oxygen about in the proportion in which they exist in water. This will cause an increase in the value of c, and one which may become appreciable, since the value of c for vapor of water is .4072.

9. *Theoretical determination of the heat of combustion.*—The heat of combustion, which, like the specific heat, is one of the characteristic elements of the force of explosive substances, may be calculated when we know:

1st. The chemical composition of the substance and the final products.

2d. The quantities of heat absorbed or given out by the formation of the compounds which make up the initial and final state of the reaction.

It results, in fact, from one of the fundamental laws of thermo-chemistry, that the amount of heat given out by a continuous series of reactions is equal to the difference between the heat of formation of the compounds in the initial and final states.

Berthelot, who was the first to state this law, basing it upon the well-known law of the conservation of energy, has made numerous and important applications of it to the heats of combustion of explosive substances. It is thus made possible, by studying the calorimetric researches of various experimenters, and, without making further determinations, to explain in a rational manner the characteristic properties of the detonating substances, which, before the utterance of this law, could be compared only by empirical methods.

We shall not dwell further upon these interesting researches, but, referring the reader to the works of that eminent chemist for further details, shall recall some experimental laws which bear upon the same subject.

9. *Experimental determination of the heat of combustion.*—To determine by experiment the heat of combustion of an explosive substance we have only to fire a known weight of it in a vessel plunged in a calorimeter, and to observe the increase of the temperature of the bath when it has absorbed the heat of the reaction.

The first determination of this nature was made by Bunsen and Schischkoff upon a powder like our sporting powder. The experiment showed that the products of combustion of a kilogram of this powder give out 644 *calories* in cooling to a temperature of 20 degrees.

We have attempted, in a series of experiments at the central depot of the State manufactory, to resume these researches, and to extend them to various substances. We shall give, further on, the results of these experiments.

Calorimetric determinations do not give all the heat of combustion, Q. The measured quantity, q, is that which is given out by the products of combustion in passing from the temperature of combustion, T, to the absolute temperature of the calorimeter t. The difference, $Q-q$, represents, then, the heat which would be given up in passing from t to absolute zero. It is generally very small, compared to Q, because of the extreme elevation of the temperature of combustion. We may, moreover, easily find its value if we know the specific heat, c, between the limits of t and zero. We would have,

$$Q = q + ct. \tag{43}$$

The specific heats corresponding to the lower temperatures are, in general, not known; for the want of more precise data, we may substitute for them the theoretical values found in No. 7. These values are too small; but the resulting error, affecting only a relatively very small term, would be small. The mean temperature of our experiments being 17 degrees, the corresponding value of the absolute temperatures is $t = 273 + 17 = 290$.

We may thus, for these elements, approximately correct the observed heats, and derive from them the corresponding heats of combustion.

The following table gives, for the principal explosives:

1st. The quantity of heat q, measured by the calorimeter;

2d. The heats of combustion Q, deduced from the foregoing by (43);

3d. The temperatures of combustion, calculated from the heats of combustion Q, and the theoretical specific heats by the formula $T = \frac{Q}{c}$.

Name of explosive.	Heat of combustion. Measured. q. Calories.	Heat of combustion. Corrected. Q. Calories.	Temperature of combustion. Degrees cent.
Best sporting powder, . . .	807	849	5870
Cannon powder, ...	553	795	5500
Fine-grained powder, called B, .	731	773	5350
Commercial powder, . . .	694	736	5090
Ordinary blasting powder, . .	570	612	4240
Chloride of nitrogen, . . .	16	339	4180
Nitro-glycerine, ...	1720	1784	8120
Gun-cotton, ...	1056	1123	4850
Picrate of potassium, . . .	787	840	4590
Mixture of 55 picrate of potassium and 45 saltpetre, ...	916	964	5810
Equal parts of picrate and chlorate of potassium, ...	1180	1224	8090

These figures are the results of calorimetric experiments made at the central depot, except those for the heat of combustion of chloride of nitrogen, which were made by Deville and Hautefeuille.

10. *Theoretical determination of the specific volume* v_0.—The specific volume of the products of combustion presents greater difficulty in its determination than any other of the elements. For we can only observe the products in their final state, and can know nothing experimentally of their condition at the instant of maximum tension. We can, however, make a calculation according to two hypotheses, corresponding to two extreme cases which comprise, probably, the truth.

We can make the calculation as though the elements of the mixture were entirely dissociated, having thus a superior limit of the possible specific volumes; and we have also an inferior limit by supposing that the compounds found in the final state are already formed, and are simply vaporized at the time of maximum tension.

First case. *Complete dissociation.*—Resuming the notation of No. 5, let us call $a_1, a_2, a_3, \ldots$ the proportions of the elements forming unit of weight of the substance, and $\omega_1, \omega_2, \omega_3, \ldots$ their molecular weights. The corresponding specific volumes are (Chapter I, No. 13)

$$v_1 = \frac{h}{\omega_1}, \quad v_2 = \frac{h}{\omega_2}, \quad v_3 = \frac{h}{\omega_3},$$

and the specific volume of the mixture is, by (23),

$$v_0 = h\left(\frac{a_1}{\omega_1} + \frac{a_2}{\omega_2} + \frac{a_3}{\omega_3} + \ldots\right) \tag{44}$$

Second case. *Vaporization of the final products.*—If the final products have been found by chemical analysis, the specific volume of each one may be calculated from its molecular weight, which, generally, is equal to its equivalent. This computation involves no other possible error than the one which results from our being obliged, in certain cases, to take the half of the equivalent. The cases in which the molecular weight is one-fourth of the equivalent are very exceptional, and may be left out of account.

After the preliminary calculation, it is only necessary to take the mean of the specific volumes of the bodies forming the final state according to the proportions of the mixture.

11. The calculation is simplified by applying first the equivalent of the explosive substance, and afterwards reducing to unit weight. If the composition in the final state is known, in equivalents, the specific volumes in the two limiting cases may be simply calculated at the same time. Suppose, for example, that the explosive is represented by the formula $\omega_1^{n_1}\,\omega_2^{n_2}\,\omega_3^{n_3}\,\ldots$; ω_1, ω_2, ω_3, being the molecular weights of the elements, so that

$$e = n_1\,\omega_1 + n_2\,\omega_2 + n_3\,\omega_3 + \ldots$$

is the equivalent of the substance.

Call Π_1, Π_2, Π_3, . . . the molecular weights of the bodies of the final state, and

$$e = N_1\,\Pi + N_2\,\Pi_2 + N_3\,\Pi_3 + \ldots$$

the formula of the reaction.

Reasoning as we have already done several times, particularly to obtain formula (42) for the specific heats, we easily find that the specific volume is represented by the formula

$$v_0 = \frac{h\,(n_1 + n_2 + n_3 + \ldots)}{e}, \tag{45}$$

in the case of *entire dissociation;* and by the formula

$$v_0 = \frac{h\,(N_1 + N_2 + N_3 + \ldots)}{e}, \tag{46}$$

in the case of the *volatilization of the compounds of the final state.* The value of h is (Chapter I, No. 13), $h = 11.15$.

12. We consider, for example, nitro-glycerine, whose combustion is represented by Berthelot by the equation

$$C^6H^5(NO^6H)^3 = 6CO^2 + 5HO + 3N + O,$$

or, showing better the molecular weights,

$$(C^2)^3H^5N^3(O^2)^9 = 6CO^2 + 5HO + 3N + \tfrac{1}{2}O^2.$$

Consequently, the specific volume is,

$$v_0 = \frac{11.15\,(3+5+3+9)}{227} = .983,$$

in the case of complete dissociation; and

$$v_0 = \frac{11.15\,(6+5+3+\frac{1}{2})}{227} = .712,$$

in the case of the vaporization of the bodies of the final state.

In the following table are written the calculated results for some explosive bodies. The formulas of reaction adopted are those used by Berthelot in his researches:

Name of explosive.	Specific volume. Of elements dissociated.	Specific volume. Of products volatilized.
Best sporting powder, ...	.654	.308
Chloride of nitrogen, ...	.370	.370
Nitro-glycerine, ...	.983	.712
Gun-cotton, ...	1.056	.680
Picrate of potassium, ...	.835	.669

13. *Intermediate states.*—We see, from the table, how great an influence the chemical state of the products at the instant of maximum tension has upon the value of the specific volume. The ratio of the two limits, which is unity for chloride of nitrogen, which always decomposes into its simple elements, varies for the different bodies between that limit and two, for the best sporting powder.

Now, between these two extreme cases, there may be intermediate states, according to the varying physical conditions of the products in the very short period between the beginning of the burning and the instant of maximum tension. It is very probable that the mode of combustion may exercise a very great influence on these circumstances; and that the conditions, for example, which develop the reaction instantly in all parts of the body, differ essentially, from this point of view, from those which cause a progressive reaction.

It is probable that we may ascribe to these causes the different effects produced by the same substance under varying conditions.

14. *Theoretical calculation of the force of an explosive substance, when totally dissociated.*—Following our examination of the circumstances on which the characteristic elements of explosive substances depend, we may lay down some considerations regarding the theoretical expression for the corresponding force.

This expression is given by formula (40½), and its value may be

found when, by the aid of the preceding principles, the values of c, Q and v_0 are known. Among the results which are thus obtained, we must notice that which relates to the complete dissociation of the products of combustion.

In this case, according to the values of c and v_0 (41) and (44), the formula reduces to

$$f = \frac{p_0}{273}\frac{hQ}{K}, \tag{47}$$

and, as the constants h and K are independent of the nature of the substance, we are led to the following consequence: *The force of an explosive substance when entirely dissociated, is proportional to its heat of combustion.*

We have, also, $h = 11.15$, and $K = 2.443$; inserting these values in (47), and putting $p_0 = 1$, we have for the expression of the force in atmospheres,

$$f = .01672 \times Q. \tag{48}$$

This is evidently a limit superior to that actually reached.

15. *Intermediate states—Equivalent combustions.*—There is reason to suppose, as has already been stated, that the chemical state of the products at the instant of maximum tension is not generally that of complete dissociation; the actual state is less simple and varies with the conditions of combustion.

Among the different reactions which these conditions may cause, we note those which present the same initial and the same final states. To these reactions the following principle, laid down by Berthelot, in his thermo-chemical researches, under the name of the "principle of the calorific equivalence of chemical transformations," applies:

> Having given a system of simple or compound bodies in a determined state; if this system undergoes any physical changes bringing it to a second state, the quantity of heat absorbed or given out depends solely upon the initial and final states of the system. It is the same, whatever may be the nature or sequence of the intermediate states.

Let us call, for brevity, those reactions which have the same initial and final states, *calorifically equivalent*, or simply *equivalent* reactions.

Observing that:

1st. Equivalent reactions have the same heat of combustion Q;

2d. The specific heat of differing equivalent reactions is the same, since, according to the hypothesis of No. 3, it depends only upon the nature and proportions of the simple elements present in the reactions.

Thus the force varies only as the specific volume v_0; and we have this consequence:

The force developed by the same explosive substance, in varying equivalent reactions or combustions, is proportional to the specific volume of the products formed at the instant of maximum tension.

By the different values of this element, variable with the state of dissociation of the products, may be explained the considerable variations of the force; even though chemical analysis of the final products and calorimetric determinations remain the same.

16. *Theoretical calculation of the force of the final products when vaporized.*—The lower limit of the force of an explosive is obtained by the aid of the value of v_0, corresponding to the final state of the products, supposed entirely gasified, but not decomposed, at the instant of maximum tension.

Making the calculation for the substances whose specific volume has already been found (No. 12), and comparing them with those which are obtained by the formula for total dissociation, we have the following table:

Name of explosive.	Force.	
	Substance dissociated.	Final state vaporized.
Sporting powder, ...	14.19	6.62
Chloride of nitrogen, ...	5.67	5.67
Nitro-glycerine, ...	29.83	21.18
Gun-cotton, ...	18.78	12.08
Picrate of potassium, ...	14.04	11.25

The linear unit adopted being the meter, the table gives, in atmospheres, the pressure developed by a kilogram of the substance detonating in a cubic meter.

17. *Approximate expression for the force of an explosive substance.*—We may obtain another expression for the force of an explosive substance. It is only approximate, but is extremely simple, and has the advantage of containing only quantities which may be determined by direct experiment, without its being necessary to have recourse to the determination by chemical analysis; which, as the foregoing considerations show, depends upon the theoretical and somewhat unreliable values of the specific volumes.

Take the general expression for the force,

$$f = \frac{1}{273}\frac{p_0 v_0 Q}{c}.$$

From equation (14), which is between the heat and specific volume of a gas, we find

$$\frac{1}{273}\frac{p_0 v_0}{c} = E\left(\frac{c'}{c} - 1\right).$$

Consequently, calling n the ratio $\frac{c'}{c}$ of the two specific heats, we have the very simple formula,

$$f = (n-1)\,EQ. \tag{49}$$

The ratio n is equal to 1.40 about for hydrogen, nitrogen, oxygen, carbonic oxide, etc. We have, consequently,

$$f = \frac{2}{5}EQ, \tag{50}$$

if these gases, or others approaching nearly the state of a perfect gas, were the only products of combustion. But, in a great many cases, these products comprise besides vapor of water and other even more complex vapors, which are liquefied or solidified after the cooling. For these compounds, n is less than 1.40, and approaches unity, in consequence of the progressive formation of the final state.

If, then, we call ε the weight of permanent gas due to the combustion of a kilogram of the explosive substance, the quantity $n-1$ may, in a first approximation, be considered as a function of ε, which, vanishing with ε, becomes $\frac{2}{5}$ for $\varepsilon = 1$. We have then, nearly, $n - 1 = \frac{2}{5}\varepsilon$; and consequently,

$$f = \frac{2}{5}EQ\varepsilon. \tag{51}$$

This formula shows that, similarly to a law of Berthelot, the force of explosive substances is nearly proportional to the product of the heat of combustion by the weight of the permanent gases produced by the explosion.

If the conditions of the explosion are such that the substance is entirely, or even partially, dissociated, so as to produce compound gases for which we would have $n = 1.40$, as would be the case for carbonic oxide, we must take $\varepsilon = 1$, and the force would be given by (50). The latter agrees, as may easily be shown, with the value (47) already obtained.

18. By means of experiments made at the central depot, ε has been determined for some substances. These together with the heats of combustion, and the corresponding forces, expressed in atmospheres, are tabulated below:

Name of explosive.	Weight of gas. ϵ. *Kil.*	Heat of combustion. Q. *Cal.*	Force. f. *Atm.*
Sporting powder, ...	.337	849	4.83
Common powder, ...	.412	795	5.53
Fine-grained powder, called B, .	.414	773	5.40
Powder of commerce, . . .	.446	736	5.54
Ordinary blasting powder, . .	.499	612	5.15
Chloride of nitrogen, . . .	1.000	339	5.72
Nitro-glycerine, ...	.800	1784	24.09
Gun-cotton, ...	.853	1123	16.16
Picrate of potassium, . . .	.740	840	10.49
55 parts of picrate of potassium and 45 of saltpetre, ...	.485	964	7.89
Equal parts of picrate and chlorate of potassium, ...	.466	1224	9.63

19. *Relative force of the explosive substances.*—It is remarkable that the five powders should have about the same force, notwithstanding differences of fabrication. This result has been confirmed by experiment, which shows that the bursting charges of the five different powders for the same shell varies only 15 to 17 grams.

The mean of the forces of powders is 5.29. This corresponds to a force of 5290 atmospheres for a kilogram of powder, detonating in a liter, that is to say, in its own volume; these results are thus in accord with those made by Captain Noble upon powder exploded in its own volume. The pressure measured by him under these conditions by means of a gauge, was 37 tons on the square inch, or 5600 atmospheres, for the F. G. powder, and about 32 tons for the R. L. G. The following table shows the relative force of the different explosives:

Name of explosive.	Relative force.
Powder containing saltpetre, ...	1.
Chloride of nitrogen, ...	1.08
Nitro-glycerine, ...	4.55
Gun-cotton, ...	3.06
Picrate of potassium, ...	1.98
55 parts picrate of potassium and 45 of saltpetre, .	1.49
Equal parts picrate and chlorate of potassium, .	1.82

20. The figures of this table appear to agree sufficiently well with the real effects of explosives, fired in a manner to produce what, in the experiments undertaken with M. Roux, we have called explosions

of the second order, that is to say, explosions produced by any other agency than a strong fulminating primer.

For example, the relative force of gun-cotton is about equal to that found by the French commission on gun-cotton (3.20), derived from the comparison of the charges just necessary to burst a shell.

Other similar experiments confirm the result for nitro-glycerine. The mean charge of powder required to rupture shell was 16 grams. In the same shells, the effect of dynamite which contained 50 per cent. of nitro-glycerine, fired by a small quantity of powder, was such that 1 part of nitro-glycerine was equal to 2 parts of powder, under these circumstances. But, as Berthelot has remarked, the heat disengaged by firing dynamite is divided between the products of the explosion of the nitro-glycerine and the inert vehicle, which latter has about the same specific heat. It results that, in such dynamite, the temperature, and consequently the force, must be lowered one-half. The force of pure nitro-glycerine should then be doubled, and be estimated at about 4. The difference between this value and that given in the table is not great. We will add that, for a mixture of picrate and nitrate of potassium, the bursting charge was found to be about 11 grams, which corresponds to a relative force of $\frac{16}{11} = 1.45$; which differs little from the theoretical value, 1.49.

21. *On the relative force of some substances, when dissociated.*—In applying formula (51) to nitro-glycerine and gun-cotton, we suppose that those of the products which are not permanent gases after the cooling, are vaporized, but not dissociated, at the instant of maximum tension. If they are, on the other hand, decomposed into gas so that the value of n is 1.40, we must use formula (50). In comparing, as before, the new values with the mean force of powder, we obtain the following relative forces:

Nitro-glycerine, ...	5.68
Gun-cotton, ...	3.58

In these new conditions, then, the force of gun-cotton becomes almost four times that of powder.

As to the figure for nitro-glycerine, it appears to have been almost reached in an experiment made by M. Roux, where about 2.7 grams of pure nitro-glycerine were fired by 1 gram of powder in an ordinary shell. In this experiment, 2.7 grams of nitro-glycerine equalled 15 grams of powder; whence the value of the relative force is 5.55, which differs little from its theoretical value, 5.68.

22. *Explosions of the first order.*—But, however great the power

of dissociated nitro-glycerine may be, it still fails to account for the effects observed when this substance is fired by violent percussion or by a charge of fulminate of mercury. It results, in fact, from our experiments, that the effect of nitro-glycerine is then at least nine times that of powder.

We have proved, by a new calorimetric experiment, that the heat disengaged under these circumstances does not differ sensibly from that due to explosions of the second order. Though this proof, made upon a small quantity of dynamite, on account of the intensity of the effects produced, has not the precision of our former determinations; it excludes the possibility of ascribing the great variation in the effects produced to variations in the heats of combustion. We may add that formula (50) being applicable to the complete dissociation of the substance, we cannot suppose that the explosion of the first order can produce new decompositions into simpler elements.

We must, then, either give up the attempt to explain the phenomena by Mariotte's and Gay-Lussac's laws, of which formula (49) is a rigid deduction; or suppose that n, the ratio of the two specific heats, acquires, at the instant of maximum tension, a greater value than 1.40; this latter value is that corresponding, in normal conditions, to simple gases, near the state of a perfect gas. If, adopting this view, we seek to determine a value of n which will make the force of nitro-glycerine nine times that of powder, according to the equation,

$$(n-1)EQ = 9 \times 5.29 \times 10333,$$

we find, taking for Q the value of nitro-glycerine, 1784,

$$n = 1.632.$$

This value of n, almost equal to $\frac{5}{3}$, reaches, then, the upper limit found theoretically for mono-atomic gases (Chapter I, No. 17).

By pushing further the deductions from this result, we would be led to explain the observed effect by supposing that, under the influence of a violent commotion there may be produced, in certain substances, an unstable molecular structure, a sort of intra-molecular dissociation, capable of destroying, during an infinitely short space of time, the atomic equilibrium, and of causing separation into more simple elements than those which are produced by the normal mechanism of the internal forces.

We do not disguise the hypothetical nature of these theories; we have thought it better to formulate them, in the absence of better explanation, in order to show that these obscure phenomena are not absolutely inexplicable in the present state of science.

23. If our explanation is correct, and if other unstable nitrogenous compounds, such as chloride of nitrogen and gun-cotton, can experience, like nitro-glycerine, this *atomic dissociation* when violently detonated, the corresponding force will be obtained by putting $n = \frac{1}{2}$ in formula (50). It is then represented by the expression

$$f = \tfrac{1}{2} EQ, \tag{52}$$

and its value is, consequently, $\frac{1}{2}$ of that which is produced in the case of *molecular dissociation.*

The relative forces calculated by this formula are

Chloride of nitrogen, ...	1.80
Nitro-glycerine, ...	9.49
Gun-cotton, ...	5.97

Under these conditions then, chloride of nitrogen will produce double the effect of gunpowder; and this result is certainly not at variance with what we know of the violence of this substance.

24. The foregoing considerations seem likely to throw some light upon the cause of the varying effects produced by explosives according to the manner of their inflammation. They constrain us to admit, for each substance, the existence of a sort of scale of pressures; the point of which reached by the explosive, when the circumstances of the combustion are progressively varied, corresponds to so many sudden variations of the law of tensions.

Nitro-glycerine offers a striking example of this. Its explosive force reduced in the proportion of 5 to 3, when the detonation ceases to be "fulminante," is still more greatly lowered when the cooling of inert bodies suppresses the dissociation of a part of the products of combustion. Do these characteristic effects of the definite explosive compounds appear in the same degree in the combustion of the mixtures which make up the ordinary gunpowders?

Although these substances have a relative stability, it is probable that they produce analogous phenomena; and it is probably to circumstances of this nature that we must ascribe the origin of the discontinuities which the discussion of Rumford's experiments has enabled us to state in the form of the law that the tension of the fluids of the powder depend on their density; and perhaps also the variations in the determination by different persons of the force of powder.

Experiment alone can solve this complex problem, whose complete solution will furnish the principal element of the dynamic theory of explosives.

CHAPTER III.

On the Work of Explosive Substances.

1. We have considered in the preceding chapter the pressure developed by the combustion of an explosive substance in a fixed and resisting envelope. We shall next study the effects produced by the expansion of the gas in a variable volume, as in a gun, from the displacement of the projectile.

At first, we shall consider only the more simple of these effects, in which, the explosion being *instantaneous*, we assume that the body is entirely gasified before the volume has appreciably altered. The case of a *progressive* combustion, in which the gases are produced during a change of volume, is much more complex, and will be made the subject of a special study.

2. *Work due to the expansion of an explosive body.*—Let unit of weight of the explosive deflagrate instantaneously in a capacity whose initial volume is v_0; call v the volume at any instant of the expansion; and let us find the work corresponding to the variation of volume $v - v_0$.

If we neglect the loss of heat due to the cooling of the walls of the envelope, the transformation is adiabatic; that is, unaccompanied by gain or loss of heat. Consequently, the work corresponding, $\mathfrak{E}$, is given by (21) of the first chapter; and we have

$$\mathfrak{E} = Ec(T_0 - T); \tag{53}$$

c being the specific heat of constant volume of the products of combustion, and T_0 and T the temperatures corresponding to the volumes v_0 and v.

$\mathfrak{E}$ may be expressed as a function of the volume when the ratio of the two specific heats of the gas remain sensibly the same during the expansion. Equation (20) shows that Tv^{n-1} is then constant.

We thus have $$Tv^{n-1} = T_0 v_0^{n-1},$$

and consequently $$T = T_0\left(\frac{v_0}{v}\right)^{n-1},$$

whence we derive the new value

$$\mathfrak{E} = Ec\,T_0\left[1 - \left(\frac{v_0}{v}\right)^{n-1}\right]. \tag{54}$$

But according to equation (40), the product $c\,T_0$ is equal to the heat of combustion Q. We may therefore write

$$\mathfrak{E} = EQ\left[1 - \left(\frac{v_0}{v}\right)^{n-1}\right], \tag{55}$$

and the work due to the combustion of any quantity ω of the substance is, evidently,

$$\mathfrak{G} = EQ\omega\left[1-\left(\frac{v_0}{v}\right)^{n-1}\right]. \tag{56}$$

3. *Maximum work or potential of an explosive substance.*—If v increases indefinitely, the limiting value of the work is, by (55),

$$h = EQ. \tag{57}$$

Consequently, *the work due to the indefinite expansion of unit of weight of a burned explosive is equal to the product of the mechanical equivalent by the heat of combustion.*

This maximum work we shall call the *potential* of the explosive substance.

It is important to notice that the expression $h = EQ$ of the maximum work that unit of weight of an explosive can produce is independent of the phenomena of dissociation; and remains the same in the case where the specific heat of the products of combustion varies during the expansion, in consequence of sudden changes in the chemical state of these products. In fact, from the equation,

$$Ec\,dT + p\,dv = 0,$$

which represents an infinitely small adiabatic transformation (Chapter I, No. 10), we derive, in all cases where the temperature varies from T_0 to absolute zero,

$$\mathfrak{G} = \int p\,dv = E\int_0^{T_0} c\,dT,$$

and the integral $\int_0^{T_0} c\,dT$ represents, in all cases, the total heat given out by the products in passing from zero to T_0; that is to say, the heat of combustion Q.

In equations (55) and (56) it is assumed, on the contrary, that the ratio n of the two specific heats remains sensibly constant during the expansion that is considered.

4. *Potentials of some explosive substances.*—The following table exhibits the values of the potentials of some powders and other explosives. These have been calculated by multiplying the experimental heats of combustion (Chapter II, No. 9) by 436. The products being divided by 1000, the figures of the table are tonnemeters.

Name of explosive.	Potential.
Sporting powder,	370
Cannon powder,	347
Fine-grained powder, called B,	337
Powder of commerce,	321
Ordinary blasting powder,	267
Chloride of nitrogen,	148
Nitro-glycerine,	778
Gun-cotton,	489
Picrate of potassium,	366
55 parts of picrate of potassium and 45 of saltpetre,	420
Equal parts of picrate and chlorate of potassium,	534

5. *Hypothesis of Bunsen and Schischkoff.*—In their chemical theory of the combustion of gunpowder, Bunsen and Schischkoff have calculated in a different way the maximum work of the powder with which they experimented. They made the calculation as though the solid residues found after the cooling remained solid throughout the phenomena and kept their initial temperature; thus evaluating only the work of the adiabatic expansion of the permanent gases.

In this hypothesis, formula (53) for the work is replaced by the following:

$$\mathfrak{G} = \varepsilon E c_1 (T_0 - T),$$

where ε is the weight and c_1 the specific heat of the *gaseous* products of the combustion of a kilogram of the substance.

But we have always $c T_0 = Q$, Q being the heat of combustion and c the specific heat of the *entirety* of the products.

We may then write

$$\mathfrak{G} = \frac{\varepsilon c_1}{c} EQ\left(1 - \frac{T}{T_0}\right),$$

and, in particular, if T tends towards zero, we have the limiting value,

$$h = \frac{\varepsilon c_1}{c} EQ.$$

According to this hypothesis, then, the value of the potential is a fraction of what we have found. For example, for sporting powder, the ratio $\frac{c_1}{c}$ differing little from unity, and ε being $\frac{1}{3}$, about, we find from the corresponding value of EQ, $h = 120$ tonne-meters, about.

This manner of calculating the work of explosive substances appears, however, hardly admissible. Bunsen and Schischkoff themselves admit that the volatilization of the solid residues at the temperature of

combustion, if not certain, is, at least, highly probable; and, if it be true, as these chemists incline to think, that the resulting vapors have a tension which may be neglected in comparison with those of the permanent gases, the heat which they give out in cooling to the mean temperature of the mixture is none the less transformed into work.

It is not reasonable that these products should keep the same temperature during the whole expansion; and if we admit, on the contrary, that, vaporized, or even solidified, they have at each instant the same temperature as the gas, we must adopt the value of the potential which we have already laid down.

In fact, in this second hypothesis the expansion of the gases ceases to be adiabatic, since they absorb the heat given out by the other products in cooling to the mean temperature of the mixture. If we call ε and $1-\varepsilon$ the relative proportions of the gaseous and non-gaseous products, and c_1 and c_2 the respective specific heats,

$$c = \varepsilon c_1 + (1-\varepsilon)c_2$$

represents the mean specific heat of the mixture.

In cooling from T_0, the initial temperature, to T, the temperature after expansion, the products give up a quantity of heat represented by $q = (1-\varepsilon)\,c_2\,(T_0 - T)$; and equation (10) applied to a weight ε of gas passing from T_0 to T, gives

$$(1-\varepsilon)\,c_2\,(T_0 - T) = \varepsilon c_1\,(T - T_0) + \frac{\mathfrak{S}}{E},$$

whence we derive

$$\mathfrak{S} = E[(1-\varepsilon)\,c_2 + \varepsilon c_1][T_0 - T],$$

that is to say—

$$\mathfrak{S} = Ec\,(T_0 - T),$$

an expression identical with (53), and leading, like it, to the value of the potential, $h = EQ$.

6. *First approximation to initial velocities.*—Formula (56) gives a first approximate determination of the initial velocity of a projectile.

If we suppose that the combustion of the powder is instantaneous, and if we neglect the loss of heat to the walls of the gun, this formula will express the work done by the gas in terms of the energy of the projectile.

We have thus—

$$\frac{m V^2}{2} = h\omega\left[1 - \left(\frac{v_0}{v}\right)^{n-1}\right], \tag{58}$$

where

V is the initial velocity,

m the mass of the projectile,

ω the weight of the charge of powder,
h the potential of the powder,
v_0 the volume of the powder chamber,
v the total capacity of the bore,
n the mean value of the ratio of the two specific heats of the products of combustion.

For this last element we may adopt, following what has already been laid down (Chapter II, No. 17), the approximate value

$$1 = n + \frac{2\varepsilon}{5}, \tag{59}$$

in which ε is the weight of gas yielded by the combustion of a kilogram of the powder used.

From (58) we have

$$V = \left(\frac{2h\omega}{m}\right)^{\frac{1}{2}} \left[1 - \left(\frac{v_0}{v}\right)^{n-1}\right]^{\frac{1}{2}}, \tag{60}$$

as the expression for the initial velocity, if the combustion is instantaneous and no heat is lost.

We shall show how this formula may be modified to take account of the progressive burning of the grains and of the loss of the heat of the products of combustion to the walls of the gun.

PART II.

EFFECTS OF POWDER IN GUNS.

CHAPTER I.

EQUATION OF MOTION OF A PROJECTILE IN THE BORE.

1. In treating this problem we shall suppose that all the products of combustion are in the state of permanent gas.

In forming the equation, certain circumstances of secondary importance will be neglected. It will be assumed, in particular, that the observed effects are exempt, or corrected for, the influences of the vent and of windage, and that the influence of rotation and resistance to forcing the projectile into the grooves has been compensated for by a proper variation of the mass of the projectile.

We shall also neglect the cooling effect of the walls of the gun, but shall return to this in a subsequent chapter.

2. From the beginning of the inflammation of the charge, the gas is formed progressively, spreads in every direction, and displaces the projectile, carrying with it the part of the charge not yet burned.

At any given instant, the density of the gas varies at different points of its mass according to an unknown law; the temperature also varies, doubtless, as the parts of gas just formed are mixed with other parts which have been cooled by expansion: for this actual state we shall substitute one in which the density and temperature are the same throughout, and equal to the mean density and temperature.

The hypothesis of a mean temperature is very nearly the same as assuming that the gaseous products during each infinitely short period of time are instantly lowered to the temperature of the gas already formed.

3. Following this hypothesis, let, at any instant t, $y = F(t) =$ the weight of the products of combustion; and $T =$ their absolute temperature.

Consider the change due to an infinitely short period of time, dt.

During the time dt, the weight y is increased by dy; and this quantity dy, in falling from the temperature of combustion T_0 to the mean temperature of the products T, gives out a quantity dq of heat, which is equal to the product of dy by c (the specific heat of constant volume) by $T_0 - T$ (the difference of temperature). Thus

$$dq = c(T_0 - T)\,dy. \tag{1}$$

In the same time, the quantity of gas already formed y, absorbs the heat dq, does the external work $d\mathfrak{G}$, and its temperature T varies dT.

Between the variations dq, $d\mathfrak{G}$, and dT, we have the thermodynamic relation,

$$dq = cy.dT + \frac{d\mathfrak{G}}{E}, \tag{2}$$

as will be apparent from (8) of Part I; recollecting that pdv is the element of external work, and that $\frac{R}{c' - c} = E$ (13).

Substituting in (2) the value of dq from (1), we have

$$\frac{d\mathfrak{G}}{E} = c(T_0 - T)\,dy - cy.dT = d[y(T_0 - T)];$$

consequently, integrating from the origin of inflammation,

$$\frac{\mathfrak{G}}{E} = cy(T_0 - T). \tag{3}$$

4. We may transform this equation by means of the relation between the temperature of a gas and its volume and pressure.

Let p = the pressure of the gas on unit surface,

V = its volume; so that

$v = \frac{V}{y}$ is the volume of unit weight.

We have then

$$pV = RyT, \tag{4}$$

R being a constant depending on the nature of the gas.

From (3) and (4) we have,

$$cpV + \frac{R\mathfrak{G}}{E} = cRT_0y. \tag{5}$$

We observe the following:

1st. The specific heat of constant pressure being c', we have

$$E = \frac{R}{c' - c},$$

hence, putting as before $n = \frac{c'}{c}$, from (5)

$$\frac{R}{E} = c\,(n-1)\,. \tag{6}$$

2d. The product of RT_0 represents, as shown by (4), the value of p corresponding to $T = T_0$, $V = 1$, and $y = 1$. It is, therefore, *the pressure developed, in constant volume, by unit weight of powder exploding in unit volume.* We shall call this quantity the *force* of the powder.

Calling this f, and recollecting (6), (5) may be written

$$pV + (n-1)\,\mathfrak{C} = fy. \tag{7}$$

Such is the relation which exists, at each instant of the expansion, between the weight, the pressure, and the external work.

5. To deduce from this the equation of motion, let

m = the mass of the projectile;

ω = its transverse section, supposed equal to that of the bore;

u = its displacement at any time t.

Neglecting the displacement of the gun during the motion of the projectile, the work $\mathfrak{C}$ will be given by the equation,

$$\mathfrak{C} = \tfrac{1}{2}\, m \left(\frac{du}{dt}\right)^2. \tag{8}$$

Also, the moving force being $p\omega$, we have

$$p\omega = m\,\frac{d^2u}{dt^2}\,. \tag{9}$$

The volume occupied by the gas V is composed of

1st. The initial volume V_0 existing around the charge before the displacement of the projectile;

2d. The volume of the interstices around the grains in the fraction of the charge which is lighted. Let $y_1 = F_1(t)$ be this volume;

3d. A volume equal to that of the powder which has burned, that is, to $\frac{y}{\delta}$, δ being the density of the powder;

4th. The cylindrical space ωu, equal to the displacement of the projectile.

We have then, $V = V_0 + y_1 + \frac{y}{\delta} + \omega u$,

or simply

$$V = \omega(u + z), \tag{10}$$

if we put, for shortness,

$$z=\frac{1}{\omega}\left(V_0+y_1+\frac{y}{\delta}\right). \tag{11}$$

We have from (9) and (10),

$$pV=m(u+z)\frac{d^2u}{dt^2}.$$

Substituting this in (7), and replacing $\mathfrak{C}$ by its value from (8), we have the following equation, which gives the motion of a projectile in a gun,

$$(u+z)\frac{d^2u}{dt^2}+\frac{n-1}{2}\left(\frac{du}{dt}\right)^2=\frac{fy}{m}. \tag{12}$$

The problem is then reduced to finding the integral of this equation; it being evident that it and the first derivative $\frac{du}{dt}$ must vanish when $t=0$.

6. If we consider the combustion instantaneous, the weight of powder burned y becomes equal to the weight of the charge ϖ, and the gas formed occupies immediately the whole of the *powder chamber*.

7. If then we call u_0 the *reduced length of the powder chamber*, that is to say, the height of an equivalent cylinder having for its base the section of the bore ω, we have $z=u_0$ in (10).

Consequently (12) becomes

$$(u+u_0)\frac{d^2u}{dt^2}+\frac{n-1}{2}\left(\frac{du}{dt}\right)^2=\frac{f\varpi}{m}: \tag{13}$$

in this form it is directly integrable.

Let $v=$ the velocity of the projectile, then we have

$$(u+u_0)v\frac{dv}{du}+\frac{n-1}{2}v^2=\frac{f\varpi}{m}, \tag{14}$$

or, separating the variables,

$$\frac{2vdv}{v^2-\frac{2f\varpi}{(n-1)m}}+(n-1)\frac{du}{u+u_0}=0,$$

and integrating,

$$\left[v^2-\frac{2f\varpi}{(n-1)m}\right](u+u_0)^{n-1}=C,$$

where C is a constant which may be determined by the condition that when $v=0$, $u=0$; hence

$$C=-\frac{2f\varpi}{(n-1)m}u_0^{n-1},$$

therefore

$$v^2 = \frac{2f\varpi}{(n-1)m}\left[1-\left(\frac{u_0}{u+u_0}\right)^{n-1}\right]. \tag{15}$$

This formula is the same as (58) of Part I. It may be put in the same form by means of (49) of Part I.

8. To suppose that the combustion is instantaneous, is not however permissible. If it were so, the gases being suddenly formed in a volume very nearly equal to that of the powder, would produce enormous pressures which the gun could not withstand. It is only by the use of *progressive powder* that the high velocities and low pressures of recent times have been attained.

We must, therefore, in (12), consider y and z as functions of t. In this case, however, the integration by ordinary methods becomes impossible.

We can obtain, however, approximate values of the integral in two extreme cases when the displacement which it represents is very great or very small compared to the reduced length of the chamber.

These two cases are, moreover, very important. The first corresponds to the ordinary cases in practice, where the chamber is generally only a rather small fraction of the volume of the bore. The examination of the second case leads to some interesting results concerning the law of tension near the origin of motion, and consequently concerning the high pressure which it is generally believed is produced at that time. The first of these cases only will be treated here.

9. Before attempting the approximate solution of (12), we may remark that it depends upon the form of the two functions $F(t)$ and $F_1(t)$, which will vary according to the manner of combustion of the charge.

We shall resume first, adding some new considerations, the notions we now have concerning these functions.

CHAPTER II.

On the Form of the Function $y = F(t)$, representing the Combustion of the Charge.

10. The exact determination of the function $F(t)$ presents great difficulties. Its form depends at the same time on the physical properties of the powder, on the form and dimensions of the charge, on the position of the vent, etc.

Piobert was the first to give its form by formulas which he had obtained under the supposition that the portion of the charge not ignited retained at each instant the form which it had before the charge was ignited, and further, that the ignition was propagated spherically with a constant velocity around each of the points of the exterior surface successively reached by the flame.

We are able to clear the problem of these restrictions and to establish, following the track of Piobert, several very complete simple formulas, rigorously determining the function $F(t)$ by the aid of two new functions $\varphi(t)$ and $\psi(t)$, defined as follows:

11. We designate by ϖ the weight of the charge of powder, $\varphi(t)$ the fraction of this charge which is in ignition at any time t reckoned from the beginning of ignition, $\psi(t)$ the fraction of one of the grains (supposed equal) of the charge which is burned after a time t, counting from the instant when the exterior surface of this grain is reached by the flame.

Then let t be any given epoch of the combustion and x any previous time; after the time x the weight of the powder in ignition is $q=\varpi\varphi(x)$. If the time x increases by dx, q receives the increase $dq=\varpi\varphi'(x)dx$.

During the time $t-x$ which elapses between the time x and t, a portion of the element dq is burned, and this portion is equal to $dq.\psi(t-x)=\varpi\varphi'(x)\psi(t-x)dx$.

The total quantity of powder burned is obtained by integrating this expression between the limits o and t, and we have

$$F(t)=\varpi\int_0^t\varphi'(x)\,\psi(t-x)\,dx. \tag{16}$$

Before making hypotheses with regard to the functions φ and ψ, we shall deduce from this formula some general properties of the function F.

12. *The function $y=F(t)$ is discontinuous.* In fact, designate:

1st, by θ the time at which all the grains are ignited;

2d, by τ the duration of the combustion of a grain.

For values of x greater than θ we have always $\varphi(x)=1$, and consequently $\varphi'(x)=0$. The corresponding elements of the integral being also o, it suffices to integrate between the limits o and θ. We have then for values of t greater than θ,

$$F(t)=\varpi\int_0^\theta\varphi'(x)\,\psi(t-x)\,dx. \tag{17}$$

If we suppose in the second place $t > \tau$, the value of $\psi(t-x)$ reduces to unity for those values of x such that $t-x > \tau$, that is to say, for such values of x as are less than $t-\tau$. It results then, for values of x between o and $t-\tau$, the element of the integral reduces to $\varphi'(x)dx$, so that we have:

$$F(t) = \varpi \int_0^{t-\tau} \varphi'(x)\, dx + \varpi \int_{t-\tau}^{t} \varphi'(x)\, \psi(t-x)\, dx,$$

that is to say,

$$F(t) = \varpi \varphi(t-\tau) + \varpi \int_{t-\tau}^{t} \varphi'(x)\, \psi(t-x)\, dx. \tag{18}$$

From formulas (16), (17) and (18) it is easy to conclude that the function $F(t)$ is generally represented by three functions, which are substituted the one for the other during the total duration of combustion. We have in fact:

1st. If $\theta < \tau$,

$$(19)\quad \begin{cases} \text{from } t = 0 \text{ to } t = \theta; & F(t) = \varpi \displaystyle\int_0^{t} \varphi'(x)\, \psi(t-x)\, dx \\ \text{from } t = \theta \text{ to } t = \tau; & F(t) = \varpi \displaystyle\int_0^{\theta} \varphi'(x)\, \psi(t-x)\, dx \\ \text{from } t = \tau \text{ to } t = \theta + \tau; & F(t) = \varpi \displaystyle\int_{t-\tau}^{\theta} \varphi'(x)\, \psi(t-x)\, dx \\ & \qquad + \varpi \varphi(t-\tau). \end{cases}$$

2d. If $\theta > \tau$,

$$(20)\quad \begin{cases} \text{from } t = 0 \text{ to } t = \tau; & F(t) = \varpi \displaystyle\int_0^{t} \varphi'(x)\, \psi(t-x)\, dx \\ \text{from } t = \tau \text{ to } t = \theta; & F(t) = \varpi \displaystyle\int_{t-\tau}^{t} \varphi'(x)\, \psi(t-x)\, dx \\ & \qquad + \varpi \varphi(t-\tau) \\ \text{from } t = \theta \text{ to } t = \theta + \tau; & F(t) = \varpi \displaystyle\int_{t-\tau}^{\theta} \varphi'(x)\, \psi(t-x)\, dx \\ & \qquad + \varpi \varphi(t-\tau): \end{cases}$$

the discontinuities corresponding to the values θ and τ of the time.

13. For each of the values θ and τ the first derivatives of two functions which are substituted the one for the other have equal values. This remarkable property of the function $F(t)$ is established without difficulty by differentiating the expressions (19) and (20) with respect to t and substituting the particular values of θ and τ in the results obtained.

It follows then, that if we represent by a curve the general form of the function $F(t)$, it will be composed of three successive arcs having a common tangent at the points of discontinuity.

14. *The case in which the duration of ignition bears a very small ratio to the duration of combustion of a grain.*

When the charge is very small or when the interstices between grains offer an easy passage to the burning gases, it is conceivable that the necessary time for all the grains to be reached by the flame is very small, and may be neglected in comparison with the total period of the combustion of a grain.

We can then suppose that all the grains are ignited simultaneously, and it follows that the combustion of the whole charge would be represented by the same function as that of the combustion of each grain. We have then, very nearly,

$$F(t) = \varpi\psi(t). \tag{21}$$

15. *The case in which the duration of ignition is very great with respect to the duration of the combustion of a grain.*

We suppose that θ is very great with respect to τ, we then can only consider the value of $F(t)$ between the limits of τ and θ, that is to say, the second of the values (20), which for values of t notably greater than τ, reduces to the following:

$$F(t) = \varpi\varphi(t). \tag{22}$$

16. *General form of the function $F(t)$.*—The functions φ and ψ can in general be developed by McLaurin's Theorem, following the powers of t; it follows then, from (21) and (22), the function $F(t)$ is of the form

$$F(t) = \varpi(at + bt^2 + ct^3 + \ldots), \tag{23}$$

when the duration of ignition of the charge is very great or very small with respect to the duration of combustion of a grain.

When these two periods are comparable the combustion operates in three distinct phases; and it is easy to establish by the aid of formulas (19) and (20), that during the first phase, the function $F(t)$ may be reduced to the form,

$$F(t) = \varpi(at^2 + bt^3 + \ldots), \tag{24}$$

that is to say, it has its first term proportional to the square of the time, if each of the functions φ and ψ have, following formula (23), their first term proportional to this variable.

It may be added that the same formula (24) may represent with a certain approximation the second phase of the combustion, because of the common value which has been shown to be true of the first and second derivatives of the two functions at the common point of discontinuity (No. 13), which are substituted the one for the other at this point.

We observe, then, that we may, in this case, represent by a continuous formula of interpolation, the whole of the combustion of the charge.

Resuming, one can readily include this formula of interpolation, as well as formulas (22) and (24), in the general form,

$$F(t) = \varpi a t^{\varepsilon}(1 + \lambda t + \mu t^2 + \ldots), \tag{25}$$

ε being an exponent equal in these limited cases to 1 or to 2, but the value of which we can leave undetermined in the calculations, it being safe to determine it later by the comparison of results obtained in theory and those of practice, in order to represent in the best possible manner the effects produced.

17. *Some particular forms of the function which represents the combustion of the grain.*

We determine very easily the form of the function by supposing, with Piobert, that all the exterior parts of the surface of a grain are ignited at the same instant, and that the combustion is propagated with a constant velocity normal to the surfaces in ignition.

We here restate the formulas for certain cases which we are to examine in the course of these researches.*

1st. *Spherical grains.*—Denoting by r the radius of the grain, and by v the velocity of combustion, we have

$$\psi(t) = 1 - \left(1 - \frac{vt}{r}\right)^3,$$

or better, calling τ the duration of the combustion of a grain, and observing that $r = v\tau$:

$$\begin{cases} \psi(t) = 1 - \left(1 - \dfrac{t}{\tau}\right)^3 \\ \psi(t) = \dfrac{3t}{\tau} - \dfrac{3t^2}{\tau^2} + \dfrac{t^3}{\tau^3} \end{cases} \tag{26}$$

* The hypothesis of the uniformity of the velocity of combustion is certainly not exact. Zero with the pressure (experiments of Bianchi), this velocity is a rapidly increasing function of the pressure of the medium in which the combustion takes place. It is conceivable, however, that we can give to it approximately a mean value, constant under certain conditions of the use of the powder, and may use this value in the application of formulas to guns when the circumstances of firing are but little different.

2d. *Cylindrical grains.*—Given h as the height of a grain and r the radius of its base, we find

$$\psi(t) = 1 - \left(1 - \frac{vt}{r}\right)^2 \left(1 - \frac{2vt}{h}\right)$$

$$(27) \qquad \psi(t) = 2v\left(\frac{1}{r} + \frac{1}{h}\right)t - \frac{v^2}{r}\left(\frac{1}{r} + \frac{4}{h}\right)t^2 + \frac{2v^3}{hr^2}t^3$$

3d. *Pierced cylindrical grains.*—We suppose the grain to be a cylinder, radius of base r, pierced at its axis by a small concentric cylindrical canal of radius r', calling h the height, we have easily:

$$(28) \qquad \begin{cases} \psi(t) = 1 - \left(1 - \dfrac{2vt}{h}\right)\left(1 - \dfrac{2vt}{r - r'}\right) \\ \psi(t) = 2v\left(\dfrac{1}{h} + \dfrac{1}{r - r'}\right)t - \dfrac{4v^2}{h(r - r')}t^2. \end{cases}$$

Formulas (26) can be applied to the normal conditions of practice; in fact, the grains, not sensibly spherical, approach nearly to a cubical form, but the formulas for spherical grains include those for a cubical form, and in general those of all grains of any polyhedral form that can be circumscribed by a sphere. With regard to formulas (27) and (28), they serve to study the effects produced by cylindrical grains, or pierced cylindrical grains (Russian prismatic powder) which have been used recently (1875) in guns of very large calibre.

18. *On the form of the function $\varphi(t)$ which represents the ignition of the charge.*—The function $\varphi(t)$ is an element whose determination offers great uncertainties. In order to find it, the hypothesis is generally made that the portion of the charge not in ignition retains at each instant the form which it had before the charge was ignited.

This hypothesis is perhaps true at the beginning of combustion, because the projectile has not been sensibly displaced, but it is probable that the charge is violently deformed by the pressure of the first gas produced, which acts unequally on the various points of its exterior surface. The grains which compose the charge are irregularly dispersed in the rapidly increasing space occupied by the products of combustion.

There is then foundation for the thought that the effects produced, particularly in long-bored guns, nearly approach those which are obtained if all the grains of the charge are ignited simultaneously.

It may, however, happen, particularly when the charge is very large and when the grains are small, that the flame communicated to

the charge is propagated gradually and slowly traverses the small interstices which separate the grains. These grains, on the other hand, burn with rapidity, and the rear of the charge, entirely burned before the flame is communicated to the whole charge, carries forward the projectile and the rest of the charge, which burns in the bore, or perhaps it does not burn until after the projectile has left the bore.

In order to evade these difficulties it is better to discard all hypotheses and to represent the law of combustion by formula (25), resorting to experiment, if necessary, to show how the coefficients $\varepsilon, \alpha, \lambda, \mu \ldots$ depend upon the conditions of firing.

19. *On the form of the function $y_1 = F_1(t)$, which represents the volume of the interstices of the portion of the charge in ignition.*

The general equation (12) of the motion of the projectile, includes, besides $F(t)$, another function of the time, designated by y_1 in the expression for z, (11), and which represents at any epoch of the combustion, the volume of the interstices of that portion of the charge which is in ignition at that epoch.

This function can be at once obtained, since we know the function $\varphi(t)$ which expresses the fraction of the charge in ignition; hence, let

d = the *gravimetric* density of the charge,

δ = the density of the grain,

$\varpi\varphi(t)$ = the weight of the ignited portion of the charge,

$\frac{\varpi\varphi(t)}{d}$ is the total volume, and

$\frac{\varpi\varphi(t)}{\delta}$ is the volume of the grains.

$$y_1 = \varpi\left(\frac{1}{d} - \frac{1}{\delta}\right)\varphi(t). \tag{29}$$

20. *On the form of the function z in the case of instantaneous ignition.*

Since the conditions of firing are such that all the grains are simultaneously ignited, the first gas produced expands instantly through the interstices of the grains, and occupies, at the beginning, a volume equal to that of the powder chamber diminished by that of the grains.

This volume should be equal to $V_0 + y_1$ from the value of z, (11), and is evidently

$$\omega u_0 - \frac{\varpi}{\delta}.$$

If we put, then, for brevity,

$$z_0 = u_0 - \frac{\varpi}{\omega\delta}, \tag{30}$$

we have in this particular case,

$$z = z_0 + \frac{y}{\omega\delta}. \tag{31}$$

21. It is to be remembered, finally, that if we call Δ the *density of loading* or the ratio of the weight of the charge to the volume of the powder chamber, we may write

$$\Delta = \frac{\varpi}{\omega u_0},$$

a relation which permits us to put formula (30) in two forms,

$$z_0 = u_0\left(1 - \frac{\Delta}{\delta}\right), \tag{32}$$

$$z_0 = \frac{\varpi}{\omega}\left(\frac{1}{\Delta} - \frac{1}{\delta}\right). \tag{33}$$

These we shall frequently use in the course of our researches.

CHAPTER III.

Approximate Integration of the Equation of Motion (12).

22. This equation is the following,

$$(u+z)\frac{d^2u}{dt^2} + \frac{n-1}{2}\left(\frac{du}{dt}\right)^2 = \frac{fy}{m}. \tag{34}$$

If we suppose that the duration of the inflammation is very small as compared with that of combustion, the function y is of the form,

$$y = \varpi at\,(1 + \lambda t + \mu t^2 + \ldots), \tag{35}$$

ϖ being the weight of the charge, and a, λ, μ constants depending upon the form and physical properties of the grains.

It may be shown that the method of integration need be slightly changed only in the general case when y is of the form (25), the exponent ϵ not being unity. In this case the function z, as well as the expression

$$V = \omega(u+z),$$

representing the volume occupied by the gas at any time, reduces (No. 20) to the expression

$$z = z_0 + \frac{y}{\omega\delta}, \tag{36}$$

z_0 having the values (32) or (33).

23. The integration will be much simplified by putting $z = z_0$. This gives too small a value to the volume occupied by the gas, since it neglects the volume of the powder which has been burned. But the resulting error will not be great, because the quantity which is neglected is infinitely small at the origin of motion, and subsequently, as the projectile moves, becomes a very small fraction of the total volume.

This simplification, without which the integration would perhaps be impossible, has then the result of giving too large a value to the pressure, and consequently to the acceleration of the projectile.

Putting $z = z_0$ in (34), replacing y by its value in (35), and writing for shortness,

$$\theta = \frac{n-1}{2}, \qquad K = \frac{f\varpi a}{m}, \tag{37}$$

it becomes,

$$(u+z_0)\frac{d^2u}{dt^2} + \theta\left(\frac{du}{dt}\right)^2 = Kl(1+\lambda t+\mu t^2+\ldots). \tag{38}$$

This is the final equation which we have to integrate, having given the initial conditions, $u = 0$, and $\frac{du}{dt} = 0$.

24. *First approximation.*—It has already been remarked that this expression does not appear to be integrable by ordinary methods. In attempting to represent it by a series arranged according to powers of t, the series is found to become rapidly divergent.

We shall, however, obtain a sufficient approximation by the following method: put $z_0 = 0$ in (38), and call u_1 those of the integrals of the equation

$$u\frac{d^2u}{dt^2} + \theta\left(\frac{du}{dt}\right)^2 = Kl(1+\lambda t+\mu t^2+\ldots), \tag{39}$$

which vanish with their first derivatives when $t = 0$.

The equation (39), corresponding to a ballistic problem in which the volume occupied by the gas is always too small by ωz_0, gives for any value of t, values $\frac{d^2u}{dt^2}$, and consequently $\frac{du}{dt}$ and u greater than those given by (38). Consequently u_1, $\frac{du_1}{dt}$, $\frac{d^2u_1}{dt^2}$, are superior limits of u, $\frac{du}{dt}$, and $\frac{d^2u}{dt^2}$.

25. It is easy to show that the difference between u and u_1 is always less than z_0. We have from (38),

$$u + z_0 = \frac{Kt(1 + \lambda t + \ldots) - \theta\left(\frac{du}{dt}\right)^2}{\frac{d^2u}{dt^2}}.$$

Putting in the second member of this equation $u = u_1$, we substitute for $\frac{du}{dt}$ and $\frac{d^2u}{dt^2}$ larger quantities, and consequently we diminish its value. Since, by (39), the second member becomes u_1, we have

$$u + z_0 > u_1,$$

and consequently, since $u_1 > u$,

$$u_1 > u > u_1 - z_0. \tag{40}$$

It results that the ratio $\frac{u}{u_1}$ approaches unity when, for increasing values of t, u becomes greater as compared with z_0. We may consequently take u_1 for the approximate expression of the displacement when this latter is considerably larger than z_0, which itself, by (32), is only a fraction of the length of the chamber.

This condition is satisfied in small arms when the projectile is near the muzzle. In great guns the capacity of the powder chamber cannot generally be considered as a very small fraction of the capacity of the bore. We shall see further on, how the solution may be modified so as to obtain a second approximation which appears to be sufficiently accurate for all purposes.

We shall first establish some important properties of the function u_1, defined by (39). It gives, it will be remembered, a first approximation to the solution in view.

26. *Integration of* (39) *by series.*—The integral u_1 of the auxiliary equation (39) may be expressed in a series of such rapid convergence that two or at most three terms suffice for purposes of application.

To satisfy this equation put

$$u_1 = At^a(1 + bt + ct^2 + \ldots), \tag{41}$$

a, b, c, . . . being constants to be determined.

The equation becomes, after substitution,

$$[a(a-1) + a^2\theta]A^2t^{2a-2}[1 + Pt + Qt^2 + \ldots] = Kt(1 + \lambda t + \mu t^2 + \ldots). \tag{42}$$

Writing, for shortness,

$$(43)\quad \begin{cases} P=\dfrac{2[a+(a+1)\theta]}{a-1+a\theta}\,b, \\ Q=\dfrac{2[(a^2+a+1)+2a(a+2)\theta]c+[a(a+1)+(a+1)^2\theta]b^2}{a(a-1)+a^2\theta}, \\ \quad\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{cases}$$

we shall have an integral by taking

$$(44)\qquad \begin{gathered} 2a-2=1,\quad [a(a-1)+a^2\theta]A^2=K, \\ P=\lambda,\quad Q=\mu. \end{gathered}$$

From the first two of these we have

$$(45)\qquad a=\frac{3}{2},\quad A=2\left[\frac{K}{3(1+3^\theta)}\right]^{\frac{1}{2}}.$$

The other equations will determine $b, c, \ldots$ Replacing a by $\frac{3}{2}$, we have

$$(46)\quad \begin{cases} \dfrac{2(3+5^\theta)}{1+3^\theta}\cdot b=\lambda, \\ \dfrac{(3^3+4^{2\theta})c+(15+25^\theta)b^2}{3(1+3^\theta)}=\mu, \\ \quad\cdots\cdots\cdots\cdots\cdots\cdots \end{cases}$$

whence we have the values,

$$(47)\quad \begin{cases} b=\dfrac{1}{2}\cdot\dfrac{1+3^\theta}{3+5^\theta}\cdot\lambda, \\ c=\dfrac{1+3^\theta}{3^3+4^{2\theta}}\cdot\left[3^\mu-\dfrac{5}{4}\cdot\dfrac{1+3^\theta}{3+5^\theta}\lambda^2\right], \\ \quad\cdots\cdots\cdots\cdots\cdots\cdots \end{cases}$$

Substituting the values of $a, A, b, c, \ldots$ in (45) and (47) in (41), that expression becomes an integral of (39). Also, this integral and its derivative vanish when $t=0$; it satisfies therefore all the conditions imposed, and may serve in the evaluation of the integral we wish to determine.

27. *Expression of t as an explicit function of u_1.*—We derive from (41) the following expression for t,

$$(48)\qquad t=\left(\frac{u_1}{A}\right)^{\frac{1}{a}}+p\left(\frac{u_1}{A}\right)^{\frac{2}{a}}+q\left(\frac{u_1}{A}\right)^{\frac{3}{a}}+\ldots$$

$p, q, \ldots$ being coefficients which may be determined by substituting in (41) the value of t (48), and equating like powers of $\frac{u_1}{A}$. We thus find

$$(49)\qquad \begin{cases} p = -\frac{1}{a}\, b \\ q = -\frac{1}{a}\, c + \frac{a+3}{2a^2}\, b^2, \\ \dots\dots\dots\dots \end{cases}$$

and, replacing a by $\frac{3}{2}$,

$$(50)\qquad \begin{cases} p = -\frac{2}{3}\, b \\ q = -\frac{2}{3}\, c + b^2 \\ \dots\dots\dots \end{cases}$$

28. *Expression of* $\frac{du_1}{dt}$ *as a function of* u_1.—Taking the derivative of u_1 with reference to t, we have

$$(51)\qquad \frac{du_1}{dt} = aAt^{a-1} + (a+1)\, bAt^a + (a+2)\, cAt^{a+1} + \dots$$

and, replacing t by its value in (48), we have

$$(52)\qquad \frac{du_1}{dt} = aA^{\frac{1}{a}} u_1^{\frac{a-1}{a}} + 2bu_1 + \left(3c - \frac{3a+1}{2a} \cdot b^2\right) A^{-\frac{1}{a}} u_1^{\frac{a+1}{a}} + \dots$$

which, writing $a = \frac{3}{2}$, becomes,

$$(53)\qquad \frac{du_1}{dt} = \frac{3}{2}\, A^{\frac{2}{3}} u_1^{\frac{1}{3}} + 2bu_1 + \left(3c - \frac{11}{6}\, b^2\right) A^{-\frac{2}{3}} u_1^{\frac{5}{3}} + \dots$$

where the values of A, b, c, ... are to be taken from (45) and (47); K will be found in (37).

We have thus a first approximation for the velocity in terms of the space passed over.

29. *Approximate expression for the temperature.*—It will be of interest to find, in the degree of approximation adopted, the law by which the mean temperature of the products varies. It would always be T_0 if the products were formed without performing external work; it varies in a gun because of the expansion and the progressive burning of the powder.

The absolute temperature T at any time is determined by (4),

$$T = \frac{pV}{Ry},$$

and from (9) and (10),

$$(54)\qquad T = \frac{m(u+z)\frac{d^2u}{dt^2}}{Ry}.$$

Now, in the approximation adopted, we neglect z in comparison with u, and replace u by u_1. We have then

$$T = \frac{m u_1 \frac{d^2 u_1}{dt^2}}{Ry},$$

or, from the values of u_1 and y in (41) and (35), and replacing a by $\frac{3}{2}$,

$$T = \frac{3}{4} \cdot \frac{A^2}{R\varpi a} \cdot \frac{1 + 6bt + \dots}{1 + \lambda t + \dots}. \tag{55}$$

We have also, from (45), $A^2 = \frac{4K}{3(1+3\theta)}$.

Finally, noting the value of K in (37), and that $f = RT_0$ (No. 4), this becomes

$$T = \frac{T_0}{1+3\theta} \cdot \frac{1 + 6bt + \dots}{1 + \lambda t + \dots}. \tag{56}$$

The terms containing powers of t higher than the first have been neglected, for shortness. By (47), b, c, ... may be expressed in terms of λ, μ, ..., the coefficients of the function y, which represents the weight of powder burned. We may write then

$$T = \frac{T_0}{1+3\theta} \chi(t),$$

$\chi(t)$ being a function which depends exclusively upon the law which governs the manner of combustion of the charge. It is clear that (56) does not represent the thermic state at the origin. From the approximation adopted to establish this formula, we see that it represents a state towards which the real state tends as the displacement increases.

30. If the combustion of the charge were uniform, (35) would reduce to its first term; and we would have $\lambda = 0$, $\mu = 0$, ..., $b = 0$, $c = 0$, ..., and consequently

$$T = \frac{T_0}{1+3\theta}.$$

This, then, is the limiting temperature, towards which the real temperature would approach in this case. In all cases, from the moment when the combustion is complete, the temperature varies according to the known law of adiabatic transformations.

31. *Second approximation.*—We shall now see how we may, in guns in which the powder chamber is not a very small fraction of the bore, obtain a more exact formula for the muzzle velocity than that which results from (53).

We derive from (38)

$$\theta\left(\frac{du}{dt}\right)^2 = Kt(1+\lambda t+\ldots)-(u+z_0)\frac{d^2u}{dt^2}.$$

Replacing u by u_1 in the second member, we diminish its value; we have therefore,

$$\theta\left(\frac{du}{dt}\right)^2 > Kt(1+\lambda t+\ldots)-(u_1+z_0)\frac{d^2u_1}{dt^2},$$

that is, from (39) which u_1 satisfies,

$$\theta\left(\frac{du}{dt}\right)^2 > \theta\left(\frac{du_1}{dt}\right)^2 - z_0\frac{d^2u_1}{dt^2}.$$

Consequently, if we determine v by the expression

$$v^2 = \left(\frac{du_1}{dt}\right)^2 - \frac{z_0}{\theta}\frac{d^2u_1}{dt^2}$$

we shall get too small a value. Writing $\frac{du_1}{dt} = v_1$, we have

$$v^2 = v_1^2 - \frac{z_0}{\theta}\frac{dv_1}{dt},$$

and, supposing the second term small compared with the first, we have the approximate value

$$v = v_1 - \frac{z_0}{2\theta v_1}\frac{dv_1}{dt}.$$

We have also,

$$\frac{dv_1}{dt} = \frac{dv_1}{du_1}\cdot\frac{du_1}{dt} = \frac{dv_1}{du_1}v_1,$$

consequently

$$v = v_1 - \frac{z_0}{2\theta}\frac{dv_1}{du_1}.$$

Substituting for v_1 its value in (53), we have

$$(59)\qquad v = \frac{3}{2}A^{\frac{2}{3}}u_1^{\frac{1}{2}}\left(1-\frac{1}{6\theta}\cdot\frac{z_0}{u_1}\right)+2b\left(u_1-\frac{z_0}{2\theta}\right)+\left(3c-\frac{11}{6}b^2\right)A^{-\frac{2}{3}}u_1\left(1-\frac{5}{3\theta}\cdot\frac{z_0}{u_1}\right)+\ldots$$

This value of v is too small; we may make an approximate correction in replacing u_1 by $u+z_0$, which is a superior limit. This gives

$$(60)\qquad v = \frac{3}{2}A^{\frac{2}{3}}(u+z_0)^{\frac{1}{2}}\left(1-\frac{1}{6\theta}\cdot\frac{z_0}{u+z_0}\right)+2b\left(u+z_0-\frac{z_0}{2\theta}\right)+\ldots$$

In its applications, we shall only use two terms of this equation, and shall neglect z_0 in the second term, which is generally only a

small fraction of the first. The formula for initial velocity then becomes

$$v = \frac{3}{2} A^{\frac{2}{3}} (u + z_0)^{\frac{1}{3}} \left(1 - \frac{1}{6\theta} \cdot \frac{z_0}{u + z_0}\right) + 2bu,$$

or

$$v = \frac{3}{2} A^{\frac{2}{3}} u^{\frac{1}{3}} \varepsilon_1 + 2bu, \tag{61}$$

by putting

$$\varepsilon_1 = \left(1 + \frac{z_0}{u}\right)^{\frac{1}{3}} \left(1 - \frac{1}{6\theta} \cdot \frac{z_0}{u + z_0}\right). \tag{62}$$

We have also, developing in powers of $\left(\frac{z_0}{u}\right)$,

$$\varepsilon_1 = 1 - \left(\frac{1}{6\theta} - \frac{1}{3}\right) \frac{z_0}{u} + \left(\frac{1}{6\theta} - \frac{1}{18\theta} - \frac{1}{9}\right) \left(\frac{z_0}{u}\right)^2 + \cdots \tag{63}$$

32. We might be led to neglect, in the above value of ε_1, the terms containing z_0^2 and those following, as has been done in substituting the approximate value (58) for the value of v in (57). But the value of ε_1 would then be too small in guns where the ratio $\frac{z_0}{u}$ has a large value.

We shall satisfy, however, experimental facts if we use (63) and the approximation of (58). It appears that the agreement of theory and practice requires that, in place of (57) where the value of v is too small, we should use (58), which is greater than the exact square root.

33. *Formulæ of initial velocities.*—It only remains, to obtain our final formula, to substitute in (61), for A and b, their values, (45) and (47). Putting then,

$$P = \left(\frac{9}{2 + 6\theta}\right)^{\frac{1}{3}}, \qquad Q = \frac{1 + 3\theta}{3 + 5\theta}, \tag{64}$$

it becomes

$$v = P \left(\frac{fa\varpi u}{m}\right)^{\frac{1}{3}} \varepsilon_1 + Q\lambda\mu; \tag{65}$$

where

f is the force of the powder,

ϖ the weight of the charge,

m the mass of the projectile,

u the space passed over by the projectile,

a, λ, are defined by the relation, $y = \varpi at\,(1 + \lambda t + \ldots)$, which represents the weight of powder burned after the time t.

ε_1 is given by (63),

z_0 is given by (32),

u_0 being the reduced length of the chamber,
Δ the density of loading,
δ the density of the grains,
P and Q are given by (64), and
$\theta = \frac{n-1}{2}$, n being the ratio of the two specific heats of the products of combustion.

34. *Influence of the mode of combustion.*—For each particular mode of combustion there is a corresponding form of (35), and consequently system of values of $a, \lambda, \ldots$, which we have only to substitute in (65) to derive the theoretical results. We shall return to this subject, only examining now the case in which the grain is a sphere, or a polyhedron in which a sphere may be inscribed.

35. *Case of spherical grains.*—When the grain is a sphere, or a polyhedron which may be inscribed in a sphere, the combustion is represented by the formula

$$y = \frac{3\varpi t}{\tau}\left(1 - \frac{t}{\tau} + \frac{t^2}{3\tau^2}\right),$$

τ being the time of burning of a grain. We have consequently

$$a = \frac{\tau}{3}, \quad \lambda = -\frac{1}{\tau}, \quad \mu = \frac{1}{3\tau^2}; \tag{66}$$

and the various formulæ deduced in this chapter are reduced to the following:

1st. Coefficients $b, c, \ldots$ of u_1. The coefficients (47) of the series (41) become

$$\begin{cases} b = -\frac{1}{2}\,\frac{1+3\theta}{3+5\theta}\cdot\frac{1}{\tau}, \\ c = \frac{1}{8}\cdot\frac{1+3\theta}{19+21\theta}\cdot\frac{7+5\theta}{3+5\theta}\cdot\frac{1}{\tau^2}. \end{cases} \tag{67}$$

These values are also necessary to calculate the coefficients of the series (50) and (53).

2d. *Formula of velocity.*—In this case, (65) becomes

$$v = 3^{\frac{1}{2}} P\left(\frac{f\varpi u}{m\tau}\right)^{\frac{1}{2}} \varepsilon_1 - Q\frac{u}{\tau};$$

it is this form that we shall frequently apply in our researches.

But before discussing the results which we have obtained, and examining how far they are in accord with experimental facts, it is necessary to correct them for the cooling effect of the walls of the gun. The next chapter will be devoted to this point.

CHAPTER IV.

On the Effect of the Cooling of the Powder Gases by the Interior Wall of the Gun.

36. Hitherto, works on interior ballistics have not taken into account, to our knowledge, the effects of the cooling of the powder gases by the walls of the bore (1875). This cooling cannot be neglected; in fact, M. de Saint Robert shows experimentally, that by firing a gun the heat absorbed by the piece is a very notable fraction, a fourth for example, of the heat developed by the combustion of the charge.

It is then inexact to suppose that the total heat of combustion is transformed into work, and we shall proceed to show how we must, in consequence, modify the general equation for the movement of the projectile which we have established in the first chapter.

37. The fundamental relation of this analysis is equation (2), which is applied, during an infinitely small period of the expansion, to the thermodynamic transformation of the gases already formed. It expresses that the heat dq absorbed by these gases is equivalent to the alteration of their sensible heat, increased by the exterior work accomplished. When there is no heat lost the quantity dq is equal to the heat set free by the gaseous products during the infinitely small period dt of the transformation. It is then given by formula (1).

But if the walls of the bore absorb heat, it becomes necessary to deduce from (1) the quantity absorbed.

This quantity may be represented by $h\sigma dt$, σ being the enveloping surface of the volume occupied by the gas, and h the velocity of flowing of the heat for a unit of surface of the walls of the bore. The following value then follows:

$$dq = c(T_0 - T)dy - h\sigma dt, \tag{69}$$

which should be substituted for formula (1), and equation (2) then becomes

$$c(T_0 - T)\,dy - h\sigma dt = cy.dT + \frac{d\mathfrak{G}}{E}, \tag{70}$$

and integrating, since $t=0$, we obtain the equation:

$$\frac{\mathfrak{G}}{E} = cy(T_0 - T) - \int_0^t h\sigma dt, \tag{71}$$

which takes the place of equation (3).

38. Introducing in (71) the pressure p and the volume V of the gas, we obtain the new equation:

$$(72) \qquad pV+(n-1)\mathfrak{C}=fy-\frac{R}{c}\int_0^t h\sigma dt,$$

which corresponds to equation (7) of the first chapter.

We now show how the equation of the movement of the projectile is deduced.

The surface σ is the total surface of a cylinder having for a height the distance $u+u_0$ of the projectile from the bottom of the bore.

The surfaces of the two bases of this cylinder may be neglected, especially in the case of very long guns, in which the cooling of the gas has most influence; calling $2r$ the diameter of bore or the calibre of the gun, we have:

$$\sigma=2\pi r(u+u_0),$$

or more simply

$$\sigma=2\pi ru,$$

in cases in which the capacity of the bore is great with respect to that of the chamber.

The coefficient h is a function of the excess of the temperature of the gases over that of the wall of the bore. It varies also with the movement of the projectile, but we can substitute for it a constant approximate mean value, consequently equation (72) should be:

$$pv+(n-1)\mathfrak{C}=fy-\frac{2\pi rRh}{c}\int_0^t udt,$$

and it suffices to operate as in No. 5 to obtain the equation of the movement of the projectile.

39. By putting as heretofore $\frac{n-1}{2}=\theta$, and observing that $\frac{R}{c}=\frac{f}{q}$, q being heat of combustion of the powder; in fact, calling T_0 and q the temperature and the heat of combustion of the powder, and f the force of the powder, we have $f=RT_0$ and $q=cT_0$, so that $\frac{R}{c}=\frac{f}{q}$.

$$(73) \qquad (u+z)\frac{d^2u}{dt^2}+\theta\left(\frac{du}{dt}\right)^2=\frac{fy}{m}-\frac{2\pi rhf}{qm}\int_0^t udt.$$

In order to integrate this equation (73) we will neglect z, following the method of approximation of No. 23 we will obtain also, corrected by taking account of the cooling, the auxiliary integral designated by

u_1, and we will afterwards deduce by the formulas of No. 30 the definite integral of the problem.

The equation (73) contains the unknown function u under the integral sign. In order to evade this difficulty, it would suffice to differentiate with respect to t, which would give a differential equation of the third order; but it is more simple to operate by successive approximation and to integrate (73) after having replaced u under the integral sign by the value u_1, which may be integrated because we neglect the additional term.

We have then to integrate the equation

$$u\frac{d^2u}{dt^2}+\theta\left(\frac{du}{dt}\right)^2=\frac{fy}{m}-\frac{2\pi rhf}{qm}\int_0^t u_1\,dt \tag{74}$$

in which the second term of the second member is still an unknown function.

40. In order to find the new integral we put

$$u=u_1+u_2,\quad \frac{d^2u}{dt^2}=\frac{d^2u_1}{dt^2}+\frac{d^2u_2}{dt^2}.$$

Substituting and reducing,

$$u_1\left(\frac{d^2u_1}{dt^2}+\frac{d^2u_2}{dt^2}\right)+u_2\left(\frac{d^2u_1}{dt^2}+\frac{d^2u_2}{dt^2}\right)+\theta\left(\frac{du_1}{dt}\right)^2+2\theta\frac{du_1}{dt}\cdot\frac{du_2}{dt}$$
$$+\theta\left(\frac{du_2}{dt}\right)^2=\frac{fy}{m}-\frac{2\pi rhf}{qm}\int_0^t u_1\,dt.$$

Neglecting the square of $\frac{du_2}{dt}$ and the product of u_2 and $\frac{d^2u_2}{dt^2}$, and recollecting the condition

$$u_1\frac{d^2u}{dt^2}+\theta\left(\frac{du_1}{dt}\right)^2=\frac{fy}{m},$$

we obtain the equation

$$u_1\frac{d^2u_2}{dt^2}+2\theta\frac{du_1}{dt}\cdot\frac{du_2}{dt}+\frac{d^2u_1}{dt^2}u_2=-\frac{2\pi rhf}{qm}\int_0^t u_1\,dt, \tag{75}$$

which is linear with respect to u_2 and its derivatives.

This equation is easily integrated when we limit the value of u_1 to the first term of its development, (41). We have then

$$u_1=At^{\frac{3}{2}}, \tag{76}$$

A having the value (45), and equation (75) becomes

$$t^2\frac{d^2u_2}{dt^2}+3\theta t\frac{du_2}{dt}+\frac{3}{4}u_2=-\frac{2}{5}\cdot\frac{2\pi rhf}{qm}t^3. \tag{77}$$

We obtain an integral which vanishes with its derivative, for $t=0$, by putting

$$u_2 = -B.\frac{2\pi rhf}{qm}t^3, \tag{78}$$

B being determined by the condition:

$$B\left(6+9^\theta+\frac{3}{4}\right)=\frac{2}{5},$$

whence

$$B=\frac{8}{45}.\frac{1}{3+4^\theta}. \tag{79}$$

Consequently the integral of equation (74) is:

$$u=u_1-B.\frac{2\pi rhf}{qm}t^3. \tag{80}$$

This is the function which must be substituted for u_1 in formula (57) in order to obtain the velocity of the projectile.

41. Also, in formula (58) we should add to v_1 the corrective term:

$$v_2=-3B.\frac{2\pi rhf}{qm}t^3,$$

which perhaps can be better expressed as a function of the space passed over, as in the case of v_1. We have from (76), $t=\left(\frac{u_1}{A}\right)^{\frac{1}{3}}$ and it follows that

$$v_2=-3B.\frac{2\pi rhf}{qm}\left(\frac{u_1}{A}\right)^{\frac{2}{3}}.$$

Replacing A by its value (45) and recollecting the value (37) of K, we have

$$v_2=-R\frac{2\pi rhf}{qm}\left(\frac{m}{f\varpi a}\right)^{\frac{2}{3}}.u_1^{\frac{1}{3}},$$

by putting for brevity

$$R=\frac{24}{45}\left(\frac{3}{4}\right)^{\frac{2}{3}}\frac{(1+3^\theta)}{3+4^\theta}. \tag{81}$$

Finally, by replacing u_1 by the quantity u, the space passed over, which differs from u_1 by a quantity less than z_0, we have:

$$v_2=-R.\frac{2\pi rhf}{qm}\left(\frac{m}{f\varpi a}\right)^{\frac{2}{3}}u^{\frac{1}{3}}.$$

42. It follows, from what precedes, that formula (65) for velocities should be replaced by the formula:

$$v=P\left(\frac{fa\varpi u}{m}\right)^{\frac{1}{3}}\varepsilon_1+Q\lambda u-R.\frac{2\pi rhf}{qm}\left(\frac{m}{f\varpi a}\right)^{\frac{2}{3}}u^{\frac{1}{3}}. \tag{82}$$

We can then combine the first and third terms after having multiplied the latter by ε_1 which alters its value very little, and which is

generally of little importance compared to that of the first term. We have thus:

$$P\left(\frac{fa\varpi u}{m}\right)^{\frac{1}{3}}\left(1-\frac{R}{P}\,\frac{2\pi rhu}{qa\varpi}\right)\varepsilon_1.$$

If then we put:

$$a=\frac{R}{P}=\frac{12}{45}\cdot\frac{1+3\theta}{3+4\theta}, \tag{83}$$

$$\varepsilon_2=1-\frac{ah}{qa}\cdot\frac{2\pi ru}{\varpi}, \tag{84}$$

we have definitely:

$$v=P\left(\frac{fa\varpi u}{m}\right)^{\frac{1}{3}}\varepsilon_1\varepsilon_2+Q\lambda u. \tag{85}$$

43. The value (84) of the correction ε_2 relating to the cooling shows that it decreases for the same powder, characterized by the given values of c, f, a, according as the quantity $\frac{2\pi ru}{\varpi}$ increases. The effect of the cooling is, consequently, greatest when the weight of the charge bears a small ratio to the interior surface of the bore, a result not difficult of proof. This effect is also modified by the law of the combustion of the charge. We suppose, for example, that the grains are nearly spherical in form. The coefficient a is then equal to $\frac{3}{\tau}$, τ being the total period of combustion, and we have

$$\varepsilon_2=1-\frac{ah}{3q}\cdot\frac{2\pi ru\tau}{\varpi}. \tag{86}$$

It follows, then, other conditions being equal, the effect of cooling is the less according as the combustion of the powder is the more rapid.

44. Formula (86) includes the specific coefficient h which represents the velocity with which the absorption of heat is effected for a unit of surface of the bore. We have no experimental data at this time (1875) which gives the value of this coefficient in the extreme conditions of temperature realized in the bore of a gun. We will attempt at the time of numerical application of the formula to deduce its value from the discussion of observed velocities.

We will confine ourselves, at this time, to the observation that its value probably varies in the same sense as the calorific conductibility of the metal in the gun, and it is fair to suppose, for example, that it is greater for bronze than for iron or steel.

Formula (85) constitutes the definite result of these researches. We will perhaps examine in subsequent chapters how it may be applied to some of the more important cases of practice.

CHAPTER V.

Applications and Numerical Verifications.

45. We shall show in this chapter the methods which, it appears, will lead most exactly to the verification of the formula for velocities which has been obtained.

Resuming this formula, referring to Nos. 32 and 40 for the meaning of the letters which it contains,

$$v = P\left(\frac{fa\varpi u}{m}\right)^{\frac{1}{2}} \varepsilon_1 \varepsilon_2 + Q\lambda u, \tag{87}$$

ε_1 and ε_2 being determined by the relations

$$\varepsilon_1 = 1 - \left(\frac{1}{6\theta} - \frac{1}{3}\right)\frac{z_0}{u} + \left(\frac{1}{6\theta} - \frac{1}{18\theta} - \frac{1}{9}\right)\left(\frac{z_0}{u}\right)^2 + \ldots \tag{88}$$

$$\varepsilon_2 = 1 - \frac{ah}{qa}\,\frac{2\pi ru}{\varpi}. \tag{89}$$

46. This system of formulæ depends, firstly, upon four characteristic elements of the powder used; namely, f, a, λ, θ. The first is the force of the powder, a and λ depend upon the mode of combustion of the charge, and θ is equal $\frac{n-1}{2}$; n being the ratio of the two specific heats of the products of combustion.

Secondly, the corrective coefficient ε_2 contains a coefficient h whose value *a priori* is unknown. We have then in all five unknown constants.

47. *Determination of* θ.—We may fix upon a value of θ which, within the narrow limits of change as to proportions and physical properties to be found in ordinary powders, will not be far from the truth.

We shall use for this the formula

$$n - 1 = \frac{2}{5}\,\epsilon \tag{90}$$

(see Part I, Chapter II, No. 17). This formula gives n for any powder, when the weight of permanent gas of a kilogram of it (ϵ) has been determined by experiment.

We have made this determination for the various powders used in France; and it results from our determinations* that the values of

* Sur la chaleur de combustion des matières explosives, par MM. Roux et Sarrau (*Comptes-rendus de l'Academie des Sciences*, 14 juillet, 1873).

ε are respectively 412 and 414, say 413 as a mean value, for cannon powder, and small arm powder called B; that is to say, for two types which will probably include all war powders.

To this value of ε corresponds the value

$$n-1=.4\times.413=.1652,$$

and consequently,

$$\theta=\frac{n-1}{2}=.0826, \text{ or about } \frac{1}{12}.$$

Such would be *a priori*, and independently of any ballistic results, the value we would give θ. But it has been found that the formulæ represent observed facts better by changing this value slightly, and taking

$$\theta=\frac{1}{11}. \tag{91}$$

We shall use this value in all the calculations which follow, and we find from it the value of ε_1 in the form,

$$\varepsilon_1=1-\frac{3}{2}\frac{z_0}{u}+\frac{10}{9}\left(\frac{z_0}{u}\right)^2. \tag{92}$$

Consequently, the numerical values of P and Q, which depend upon θ, are found to be from (64),

$$P=1.5234, \qquad Q=.3684. \tag{93}$$

48. This determination, whose exactness will be further confirmed in the sequel, being made, we shall show how the formula may be tested by means of initial velocities taken in varying conditions, but with *the same powder*.

Putting then,

$$P(fa)^{\frac{1}{2}}=A,\ Q\lambda=-B,\ \frac{\pi ah}{qa}=\gamma, \tag{94}$$

$$\frac{z_0}{u}=\frac{u_0}{u}\left(1-\frac{\Delta}{\delta}\right)=x,$$

the formula we wish to verify is the following,

$$v=A\left(\frac{\varpi u}{m}\right)^{\frac{1}{2}}\varepsilon_1\varepsilon_2-Bu,$$

ε_1 and ε_2 having the values

$$\begin{cases}\varepsilon_1=1-\frac{3}{2}x+\frac{10}{9}x^2,\\ \varepsilon_2=1-\gamma.\frac{2ru}{\varpi},\end{cases} \tag{96}$$

and we should be able, if the formula is exact, to determine a system of values for A, B, and γ such that they will give correct velocities

when we vary ϖ, u, m, . . . , upon which they depend, through large limits.

49. *Experiments which have been used in the verification of the formulæ.*—We have used the velocities given by the naval artillery commission, in the numerous series of experiments, with various conditions of loading, carried out by them at Gavre with guns of the model of 1870.

The total number of measured results which have been used in the determination of the coefficients is 24; they were all taken from the official publication in the *Memorial*, and are cited in the recapitulative table which follows.

50. *Description of the powder used.*—In all these experiments, Wetteren powder, of grains of 13 to 16 millimeters, was used. The proportions are the standard ones for war powder:

Saltpetre, 75; sulphur, 12.5; charcoal, 12.5.

Process of fabrication, *tonnes et meules;* number of grains to the kilogram, about 350. Density of grain, about 1.800.

The obligation which we have been under, of using samples of powder from various sources not having absolutely the same ballistic value, has necessarily caused some variations in the results.

Name of Gun.	(1) $2r$ Met.	(2) u Met.	(3) u_0 Met.	(4) β Kil.	(5) ϖ Kil.	(6) Δ	(7) v Met.
24 cent., model 1870, No. 24,	0.242	3.445	.763	144.	12.	0.342	253.3
					16.	0.456	306.6
					20.	0.570	350.2
					24.	0.684	393.2
					28.	0.798	432.0
24 cent., model 1870, No. 24,	0.242	3.445	.763	120.	12.	0.342	273.0
					16.	0.456	332.3
					20.	0.570	379.5
					24.	0.684	427.2
					28.	0.798	467.4
24 cent., model 1870, No. 24,	0.242	3.445	.763	96.	12.	0.342	300.1
					16.	0.456	363.0
					20.	0.570	420.6
					24.	0.684	469.2
					28.	0.798	512.5
14 cent., model 1870, No. 1,	0.140	2.761	0.205	18.65	2.0	0.637	322.0
					2.5	0.797	372.9
					3.0	0.956	419.6

Name of Gun.	(1) $2r$ Met.	(2) u Met.	(3) u_0 Met.	(4) p Kil.	(5) ϖ Kil.	(6) Δ	(7) v Met.
14 cent., model 1870, No. 4,	0.140	2.731	0.239	18.65	2.5	0.684	362.0
					3.0	0.820	404.6
					3.5	0.957	444.0
14 cent., model 1870, No. 2,	0.140	2.697	0.273	18.65	3.0	0.719	391.9
					3.5	0.838	431.2
					4.0	0.958	472.0

51. Legend of the table:

$2r$ is diameter of the bore;

u initial distance of the base of the projectile from the muzzle;

u_0 reduced length of the powder chamber, or length of a cylinder of equal volume having the right section of the bore as base;

p the weight of the projectile;

ϖ the weight of the charge;

Δ the density of loading, or ratio of the weight in kilograms of the charge to the volume in cubic decimeters of the charge;

v the initial velocity.

52. *Order followed in the determination of the constants.*—We shall determine the constants in the following order:

1st. B from observed velocities in the 24 cent. gun with different weights of projectiles.

2d. The coefficient of cooling γ from the nine velocities of the three guns of 14 cent.

3d. A from all the measured velocities.

53. *Determination of B.*—If we suppose that the mass of the projectile m is the only variable, the velocity is a function of it of the form

$$v = ax - b,$$

putting $\frac{1}{m^{\frac{1}{3}}} = x$. To determine b we must have then two corresponding values of v and m. From the two equations,

$$v_1 = ax_1 - b,$$
$$v_2 = ax_2 - b,$$

we have

$$a = \frac{v_1 - v_2}{x_1 - x_2},$$

and b may then be found from either of the preceding equations.

But the use of two observations only would not give close accuracy, for the variations of the weight of the projectile are small. It results, therefore, that the denominator of a is generally very small, which in-

creases an error in the numerator. It therefore becomes necessary to use a larger number of observations. We have proceeded as follows:

For the same charge of powder, and consequently for the same value of a, let

$$v_1, v_2, v_3,$$

be three velocities corresponding to three values of the mass of the projectile. Let the corresponding values of $\left(\frac{1}{m}\right)^{\frac{1}{3}}$ be x_1, x_2, x_3. We have then the equations,

$$(1) \quad v_1 = ax_1 - b,$$
$$(2) \quad v_2 = ax_2 - b,$$
$$(3) \quad v_3 = ax_3 - b,$$

From (1) and (2), and (2) and (3), we find two approximate values $a_{1,2}$ and $a_{2,3}$ of a; and we adopt the mean value

$$a = \frac{a_{1,2} + a_{2,3}}{2},$$

and from this find for b the three nearly identical values,

$$b_1 = ax_1 - v_1,$$
$$b_2 = ax_2 - v_2,$$
$$b_3 = ax_3 - v_3,$$

and take the mean of these. Operating in this manner with the velocities of the gun of 24 cent. with the same charge of powder, the following table of corresponding values of ϖ, a, b, is formed:

ϖ Kil.	a	b
28	1.365	125.4
24	1.291	133.9
20	1.166	123.8

The mean of the values of b is 127.7. Consequently, since $b = Bu$, and the value of u for the 24 cent. gun is 3.445, we have

$$(97) \qquad B = 37.07$$

54. *Determination of* γ.—The determination of B being made by the 24 cent. gun, which alone was fired with sufficient variation in the weight of the projectile; γ was calculated from the measured velocities with the 14 cent. gun, where the influence of the cooling was relatively greater.

To this end, values were calculated for the three guns.

1st. For Bu, which were respectively,

No. 1, ...	102.3
No. 4, ...	101.2
No. 2, ...	100.0

2d. The quantities W, obtained by adding the measured velocities to the corresponding values of Bu. These will be found in the following table:

ϖ Kil.	No. 1.	No. 4.	No. 2.
2.0	424.3		
2.5	475.2	463.2	
3.0	521.9	505.8	491.9
3.5		545.2	531.2
4.			572.0

3d. The values of $\frac{W}{\varepsilon_1}\left(\frac{m}{\varpi u}\right)^{\frac{1}{3}}$, which represent, by formula (95), the product of the constant A by the corrective coefficient ε_2. The following gives the results:

ϖ Kil.	No. 1.	No. 4.	No. 2.
2.0	319.7		
2.5	329.4	328.2	
3.0	336.8	334.0	332.6
3.5		338.5	337.7
4.0			344.3

The last table gives rise to an important remark. The figures on the same horizontal line are very nearly equal; in the one above, however, this is not the case, in consequence of the fact that, though the charge was the same, the values of the length of the charge and the density of loading were different.

It results from this that the factor ε_1, which takes account of these variations, represents exactly the influence which they have upon the velocity.

The differences in the same vertical column may therefore be supposed due to the cooling, whose effect would therefore be considerable in the gun of 14 cent. We see, also, that these differences are less with large charges, as should be the case: in fact, if we suppose that the figures of the table may be represented by the product $A\varepsilon_2$, A being a constant, and ε_2 a function of the form

$$\varepsilon_2 = 1 - \gamma \frac{2ru}{\varpi},$$

we have found that by taking

$$\gamma = 0.65,$$

the quotients of the figures of the table by the corresponding values of ε_2 are sensibly constant. This results from an examination of the following table, which contains the nine quotients giving as many approximate values of A:

ϖ Kil.	No. 1.	No. 4.	No. 2.
2.0	365.7		
2.5	366.1	364.5	
3.0	367.5	364.1	362.2
3.5		364.3	363.2
4.0			366.8
Means	366.4	364.3	364.1

55. *Determination of A.*—The three values of A determined from the velocities of the gun of 14 cent. agree very closely. It is important to see whether we obtain an identical value, or a very nearly equal one, by applying the same operations to the 15 velocities taken with the 24 cent. gun.

Consequently, the value of Bu, about 127.7, has been added to these, and the results W multiplied by $\frac{1}{\varepsilon_1 \varepsilon_2}\left(\frac{m}{\varpi u}\right)^{\frac{1}{3}}$; using, for the calculation of ε_2, the coefficient $\gamma = 0.65$, already found.

These products, which give corresponding values of A, are entered in the following table:

Weight of Charge ϖ; Kil.	Projectile. 144 Kil.	120 Kil.	96 Kil.
12	368.0	364.3	367.7
16	369.3	368.1	364.5
20	367.1	366.7	368.0
24	367.3	368.2	367.7
28	366.0	366.3	365.8
Means	367.5	366.7	366.7

The general mean, 367.0, differs very little from the values obtained with the 14 cent. gun.

For A, we have then a series of four values:

24 cent. gun,	No.	24,	. . . . .		367.0
14 "	"	"	1,	. . . . .	366.4
14 "	"	"	4,	. . . . .	364.4
14 "	"	"	2,	. . . . .	364.1

whose mean is

$$A = 365.5 \tag{98}$$

56. *Resumé and formula for the Wetteren powder (grains of 13 to 16 mill.).*—We have then, for the powder used, the following coefficients:

$$A = 365.5$$
$$B = 37.07$$
$$\gamma = 0.65$$

whence the final formula,

$$V = 365.5\left(\frac{\varpi u}{m}\right)^{\frac{1}{3}}\varepsilon_1\varepsilon_2 - 37.07u. \tag{99}$$

and

$$\begin{cases} \varepsilon_1 = 1 - \frac{3}{2}x + \frac{10}{9}x^2, \\ x = \frac{u_0}{u}\left(1 - \frac{\Delta}{1.8}\right), \\ \varepsilon_2 = 1 - 0.65\frac{2ru}{\varpi}. \end{cases} \tag{100}$$

In the following table will be found a number of velocities calculated by this formula, together with the observed velocities. The table comprises, besides the 25 velocities which were used in the determination of the constants, the calculated and measured velocities in 35 other cases; namely,

1st. 18 with the gun of 14 cent. of various lengths, and various charges and projectiles;

2d. 7 of the 19 cent. gun with various charges and projectiles;

3d. 5 with the 24 cent. gun (No. 24) with cylindrical projectile of 144 kilos;

4th. 5 with the steel 24 cent. gun.

The observed velocities were all taken from the *Memorial*. There are in all 16 different conditions of fire, and the mean error of the calculated velocities is 2.8 meters.

57. Table giving calculated and corresponding measured velocities:

Name of Gun.	(1) $2r$ Meters.	(2) w Meters.	(3) w_0 Meters.	(4) p Kilos.	(5) m Kilos.	(6) Δ	(7) Velocities Measured. Meters.	(7) Velocities Calculated. Meters.	(7) Velocities Differences.
14 cent. model 1870, No. 1.	0.140	2.761	0.205	18.65	2.0	0.637	322.0	322.0	0
					2.5	0.797	372.	372.9	0
					3.0	0.956	419.6	417.0	+ 2.6
14 cent. model 1870, No. 4.	0.140	2.731	0.239	18.65	2.5	0.684	362.0	363.5	— 1.5
					3.0	0.820	404.6	406.8	— 2.2
					3.5	0.957	444.0	446.1	— 2.1
14 cent. model 1870, No. 2.	0.140	2.697	0.273	18.65	3.0	0.719	391.9	396.6	— 4.7
					3.5	0.838	431.2	434.9	— 3.7
					4.0	0.958	472.0	470.3	+ 1.7
14 cent. model 1870, No. 1, transformé.	0.140	2.560	0.273	18.65	3.0	0.719	387.9	393.1	— 5.2
					3.5	0.838	426.6	430.5	— 3.9
					4.0	0.958	466.6	465.4	+ 1.2
14 cent. model 1870, No. 1, transformé.	0.140	2.420	0.273	18.65	3.0	0.719	383.5	388.7	— 5.2
					3.5	0.838	424.0	425.4	— 1.4
					4.0	0.958	462.9	459.5	+ 3.4
14 cent. model 1870, No. 1, transformé.	0.140	2.281	0.273	18.65	3.0	0.719	378.4	383.8	— 5.4
					3.5	0.838	413.7	419.6	— 5.9
					4.0	0.958	458.3	453.2	+ 5.1
14 cent. model 1870, No. 1, transformé.	0.140	2.143	0.273	18.65	3.0	0.719	375.9	378.1	— 2.2
					3.5	0.838	414.6	413.3	+ 1.3
					4.0	0.958	456.2	446.1	+ 10.1
14 cent. model 1870, No. 2, transformé.	0.140	2.697	0.273	20.9	3.0	0.719	378.8	378.1	+ 0.7
					3.5	0.818	414.6	415.0	— 0.4
					4.0	0.938	447.4	449.0	— 1.6
14 cent. model 1871, No. 2.	0.140	2.697	0.273	23.3	3.0	0.719	366.1	361.1	+ 5.0
					3.5	0.818	400.5	396.7	+ 3.8
					4.0	0.938	428.0	429.5	— 1.5
19 cent. model 1870, No. 1.	0.194	3.002	0.512	62.5	10.0	0.661	393.3	390.6	+ 2.7
					12.5	0.826	444.6	446.4	— 2.2
					15.0	0.991	492.2	498.4	— 6.2
19 cent. model 1870, No. 1.	0.194	3.002	0.512	76.0	12.5	0.826	411.8	411.2	+ 0.6
					13.5	0.892	428.7	431.0	— 2.3
					14.5	0.958	446.4	450.4	— 4.0
					15.0	0.991	453.5	459.9	— 6.4
24 cent. model 1870, No. 24.	0.242	3.445	0.763	144. ogival	12.0	0.342	253.3	250.9	+ 2.4
					16.0	0.456	306.6	302.3	+ 4.3
					20.0	0.570	350.2	348.3	+ 1.9
					24.0	0.684	393.2	390.6	+ 2.3
					28.0	0.798	432.0	431.4	+ 0.6
24 cent. model 1870, No. 24.	0.242	3.445	0.763	144. cylindr.	12.0	0.342	253.9	250.9	+ 3.0
					16.	0.456	303.4	302.3	+ 1.1
					20.	0.570	350.0	348.3	+ 1.7
					24.	0.684	390.5	390.9	— 0.4
					28.	0.798	428.8	431.4	— 2.6
24 cent. model 1870, No. 24.	0.242	3.445	0.763	120.	12.	0.342	273.0	274.6	— 1.6
					16.	0.456	332.3	329.3	+ 3.0
					20.	0.570	379.5	378.1	+ 1.4
					24.	0.684	427.2	423.4	+ 3.8
					28.	0.798	467.4	466.5	+ 0.9
24 cent. model 1870, No. 1.	0.242	3.445	0.763	96.	12.	0.342	300.1	305.6	— 5.5
					16.	0.456	363.0	364.6	— 1.6
					20.	0.570	420.6	417.2	+ 3.4
					24.	0.684	469.2	465.9	+ 3.3
					28.	0.798	512.5	512.4	+ 0.1
24 cent. (steel), No. 1	0.242	4.036	0.796	144. cylindr.	12.	0.333	262.7	263.4	— 0.7
					16.	0.444	312.3	314.5	— 2.2
					20.	0.556	360.4	363.1	— 2.7
					24.	0.667	396.6	407.7	— 11.1
					28.	0.778	434.9	449.8	— 14.9

58. *Force and time of combustion.*—Suppose that the grains of a charge may, on the average, be considered spheres; then (68) becomes applicable, and the cofficients A and B become

$$A = 3^{\frac{1}{3}} P \left(\frac{f}{\tau}\right)^{\frac{1}{3}}, \quad B = \frac{Q}{\tau}; \tag{101}$$

and the numerical values of A and B having been already determined, we may find f and τ by the relations

$$\tau = \frac{Q}{B}, \quad f = \frac{\tau}{3}\left(\frac{A}{P}\right)^3. \tag{102}$$

Taking $A = 365.5$, $B = 37.07$, with the values of P and Q from (93) we have

$$\tau = 0.00994, \quad f = 45800.$$

These values are only a first approximation; and, although the question has no great practical importance, we shall find a closer approximation by taking account of the third term of (53).

59. *Influence of the third term of the formula for velocities.*—The term is as follows:

$$\left(3c - \frac{11}{6} b^2\right) A^{-\frac{3}{2}} u^{\frac{5}{2}}.$$

Replacing c and b by their values (67), A by its value (45), and taking $\theta = \frac{1}{11}$, we find

$$-R\left(\frac{m}{f\varpi}\right)^{\frac{1}{3}}\left(\frac{u}{\tau}\right)^{\frac{5}{3}}, \tag{103}$$

R being equal to about .05.

The following table contains the calculation of this term for the various cases with the 14 cent. gun:

ϖ Kil.	No. 1	No. 4	No. 2
2.0	15.6		
2.5	14.4	14.2	
3.0	13.6	13.4	13.1
3.5		12.7	12.5
4.0			11.9

For the 24 cent. gun (projectile of 144 kilos), with charges of

12, 16, 20, 24, 28 kilos,

we find

18.5 19.5 20.7 22.3 24.6

These values cannot be neglected, but their variations in the same gun may be nearly; because, in practice, the ratio $\frac{p}{\varpi}$ of the weight

of the projectile to that of the charge varies very little, and we may therefore replace it by a mean value.

Taking, for example, $\frac{p}{\varpi} = 4.5$, and calculating the corresponding value of $\frac{m}{\varpi} = \frac{p}{\varpi g}$, we reduce the third term to the value,

$$-R\left(\frac{u}{\tau}\right)^{\frac{5}{3}},$$

R being equal to .00108.

60. *Corrected values of f and τ for Wetteren powder.*—Consequently, for spherical grains, writing the third term, the formula for velocity becomes,

$$v = A\left(\frac{f\varpi u}{m\tau}\right)^{\frac{1}{3}}\varepsilon_1\varepsilon_2 - Q\frac{u}{\tau} - R\left(\frac{u}{\tau}\right)^{\frac{5}{3}}; \tag{104}$$

the relation $\frac{Qu}{\tau} = Bu$, which previously was used in calculating τ, is now replaced by

$$Bu = \frac{Qu}{\tau} + R\left(\frac{u}{\tau}\right)^{\frac{5}{3}}. \tag{105}$$

For the 24 cent. gun we have found $Bu = 127.7$. Substituting in the preceding equation, we find, by successive approximations,

$$\tau = 0.0116, \quad f = 53600.$$

61. This constant f is the force of the powder, that is to say the pressure on unit surface when unit weight is burned in unit volume. We have given (Part I, Chapter II, No. 17) a theoretical determination of this quantity. We have:

$$f = \frac{2}{5}EQ\varepsilon, \tag{106}$$

where

E is the mechanical equivalent of heat;

Q the heat of combustion;

ε the weight of permanent gas given by the combustion of unit weight of powder.

In the place cited, the forces of the various powders made in France are given, and, having stated (what is confirmed by experiment) that these forces are about the same, notwithstanding the differences in proportions and in manufacture, we concluded that the mean value of this element may be fixed at 5290 atmospheres for a kilogram detonating in a liter. Taking the kilogram and meter for units, this corresponds to

$$f = 54600;$$

this value does not sensibly differ from the one which results from an analysis of observed facts with guns.

This close agreement is worthy of attention; it will perhaps be considered sufficiently close to be accepted as showing that ballistic results, far from being in contradiction to thermodynamic theory, give additional strength to that theory.

62. *Determination of the velocity of cooling.*—The meaning of the coefficient γ, which has been found equal to 0.65, is given by (93),

$$\gamma = \frac{\pi a h}{q a}$$

a having the value in (83). For spherical grains, $a = \frac{3}{\tau}$, and consequently

$$\gamma = \frac{\pi a \tau h}{3q}.$$

We have then

$$h = \frac{3q\gamma}{\pi a \tau},$$

for the velocity of cooling by the walls of the gun corresponding to the mean temperature of the powder gas. We have

$$\tau = 0.0116, \qquad q = 784,^{*}$$
$$a = 0.10,^{\dagger} \qquad \gamma = 0.65,$$

and hence

$$h = 420000 \text{ about.}$$

This is the flow, in calories, per second per square meter.

This number is very large; but it does not seem inadmissible when we recollect the high temperature of the gas, and the physical laws which govern the cooling.

Although there are no experiments which would enable us to fix with precision the thermic changes, we may yet obtain an approximate idea by applying Dulong and Petit's law to recent experimental results, and particularly to those obtained by Mr. Donald McFarlane,[‡] in a series of researches intended to establish the conductibility of bodies in absolute units. He measured, in calories, and for unit surface, the velocity of cooling of a copper sphere in damp air. Twelve determinations were made, for increments of 5°, from 5 to 60 degrees centigrade.

*This number is the mean of those determined by experiment for cannon powder, and powder called B (See Part I, Chapter II, No. 18).

†This value results from (83), by putting $a = \frac{1}{11}$.

‡Proceedings of the Royal Society, Vol. XX, p. 90.

These limits are narrow; but if we admit, following Dulong and Petit, that the velocity of cooling corresponding to a difference θ of temperature is represented by a formula of the form

$$h = M(a^{\theta} - 1),$$

(M being a constant depending upon the circumstances of the cooling, and a a constant always equal to 1.0077), we may determine M from the results quoted.

Taking for units the meter and second, they are very well represented by the formula

$$h = 241\,(a^{\theta} - 1),$$

which may be taken as

$$h = 241 a^{\theta},$$

when θ is large.

In these conditions, if $\theta = 973$, the value of h becomes the same as has been found.

63. It is important also to notice that, when θ varies, the corresponding variations of h are large. Consequently, the *mean value* used in the approximate evaluation of the effect of cooling in a gun should vary with the mean temperature produced in the gun. We have taken it the same in guns of different caliber, because the conditions of fire do not differ much, and because this element appears only in a relatively small term.

On the contrary, in small arms, the surface of cooling is relatively much larger than in great guns; and the temperature must therefore be lowered more. Consequently, the value of h in this case may be much smaller; and it should be made the object of experiment.

64. We conclude, from what precedes, that the formulæ deduced from thermodynamic theories, represent the actual effects of powder more accurately and throughout a greater range than do empirical formulæ.

We shall make, in the future, new applications to field guns and small arms. We shall also show how the characteristic elements of a powder may be deduced from a small number of experiments.

In a Third Part we shall take up the study of the circumstances upon which variations of pressures in guns depend.

PART III.

NEW RESEARCHES ON THE EFFECTS OF POWDER.

PREFACE.

I. In a memoir printed in 1875, I attempted to establish formulæ giving the initial velocity of a projectile as an explicit function of the elements of fire, basing the deduction upon the theory of thermodynamics. The results which I obtained appear to agree with facts; but, in consequence of certain restrictive hypotheses, they give only an approximation with which it is unsatisfactory to conclude, especially when we desire to study the law of the variation of pressure.

II. I assumed, to establish the equation of motion:

1st. That all the products of combustion of the powder were in the state of gas or vapor.

2d. That we may approximately substitute for the actual condition of these products a condition in which all parts of the gaseous mass have a temperature and density equal to the mean of the mixture.

3d. That the velocity of combustion of the powder is constant during the phenomena.

Thus from the first two hypotheses I have calculated the pressure on the projectile and the work done, by applying the laws of Mariotte and Gay-Lussac to the entire products of combustion, as if they were in the *static* state. If we admit the third hypothesis, the law of the combustion of a grain is given by formulæ similar to those of Piobert.

III. Since my memoir was prepared, Noble and Abel have published their important researches upon the combustion of powder in a closed vessel. These seem to indicate that a part of the products of combustion are in the solid state, in the ordinary conditions of their formation. Moreover, these products are in *motion* during the entire phenomenon, and it is necessary, in evaluating the external work, to take account of these internal motions, which will vary the density and temperature throughout the gaseous mass.

Consequently, the first two hypotheses seem to be insufficient. The third is certainly inexact; and, although we may admit, as an approxi-

mation, a value of the velocity of combustion which is considerably greater than that in free air, yet it is clear that, in a rigid analysis of the effects of powder in a gun, we must take account of the law by which the velocity of combustion varies with the pressure.

Finally, the method adopted, in my last memoir, to integrate the equation of motion leaves some doubt as to the degree of precision obtained; it is also subject to conditions which make it inapplicable to points near the origin of motion; and we therefore cannot deduce from it the law of pressures at the point where these latter are highest.

IV. For these reasons I have thought it would be of interest to undertake again the solution of the questions I have already treated, correcting and completing the data wherever possible. I have been able also, at the same time, to render the analytical considerations by which I deduce from the theory its various consequences more simple and rigid. These new researches will reproduce some results already reached, but they also furnish others which seem likely to throw some light on the complex subject we have in hand.

V. After discussing, in the first chapter, the results of the last experiments of Noble and Abel, I examine in Chapter II how the general equation may be modified so as to include the results of these experiments.

This modification may be made either by supposing, as do Noble and Abel, that the solid and gaseous products are always in equilibrium as to temperature; or, following Bunsen and Schischkoff, that the solid products continue at the temperature of combustion, and that the heat of the gaseous parts only is transformed into work.

In each hypothesis, the form of the fundamental equation is the same; the values of the coefficients changing, however. The same equation also holds when we take account of the internal motions, whose disturbing influence, evaluated by the new theory of gases, would appear to have a less value than some authors suppose.

This equation, from which flow the consequences which govern the velocities and pressures, is the one obtained by M. Resal in his *Researches on the Motion of Projectiles.*

Chapter III treats of the combustion of powder under variable pressure. The law connecting the velocity of combustion and pressure is not yet known. If we suppose, however, as a first approximation, that the velocity is proportional to a positive power of the pressure, we may represent the combustion of the charge in the variable

pressure of the expansion analytically, and thus establish the equation, having taken account of all the conditions.

Chapter IV is devoted to the integration of this equation. The integration is effected by series, by the aid of certain functions defined by a system of differential equations. These functions I have called *auxiliary;* they are purely numerical and entirely independent of the elements of fire, and tables of their values similar to those of logarithms may be computed.

In the fifth and last chapter, the methods of making the calculations will be presented.

VI. I thus obtain new formulæ giving the velocity and pressures, and I deduce from these the laws by which these quantities depend, not only upon the various elements of fire, but also upon the nature of the powder and the form of grains.

Among the results obtained I will mention the calculation of the maximum pressure produced in a gun from the following data, whose determination can all be made in a laboratory:

1st. The heat of combustion of the powder.

2d. The volume of permanent gas which is produced.

3d. The velocity of combustion in free air.

The values which we shall thus reach agree exactly with those given by direct measurement by pressure gauges.

CHAPTER I.

On the Pressure produced by the Powder Gases in a Vessel whose Capacity remains constant.

1. MM. Noble and Abel have recently communicated (1876) to the Academy of Sciences, the summary of important researches on the combustion of powder. Admitting the results of these learned experimenters, that which relates to the pressures developed by the combustion of the powder in a closed vessel is particularly remarkable. It results, in fact, from the measure of the pressures that one is able to explain the facts observed, by supposing:

1st. That a part of the products of combustion is in a solid state.

2d. That the pressure due to permanent gases only can be calculated after the law of Mariotte, by deducting from the volume of the envelope that of the solid residue.

2. The calculation of the pressures, under this hypothesis, is done in the following manner: Let

C = the volume of the vessel or place of combustion.

ϖ = the weight of the powder burned.

T_0 = the absolute temperature of the products of combustion.

a = the volume at this temperature, of the solid residue furnished by the combustion of one kilogram of powder.

v_0 = the volume (at zero centigrade, and under the normal atmospheric pressure), of the permanent gases of a kilogram of powder.

The permanent gases produced by the charge ϖ of powder at T_0 occupies a volume equal to $C - a\varpi$. In order to find the corresponding pressure, it is necessary to apply the laws of Mariotte and Gay-Lussac, observing that the same gases under the normal pressure p_0 and at 273° of absolute temperature occupy a volume equal to ϖv_0; we have then

$$\frac{p}{p_0} = \frac{\varpi v_0}{C - a\varpi} \cdot \frac{T_0}{273},$$

or better,

$$(1) \qquad p = f \frac{\varpi}{C - a\varpi},$$

putting for brevity,

$$(2) \qquad f = \frac{p_0 v_0 T_0}{273}.$$

We can also introduce the *density of loading*, that is to say, the ratio $\Delta = \frac{\varpi}{C}$ of the weight of the powder to the capacity of envelope, and write,

$$(3) \qquad p = \frac{f\Delta}{1 - a\Delta}.$$

3. *Force of the powder.*—The quantity f defined by the relation (2) represents the pressure of the permanent gases of a kilogram of powder, occupying, at the temperature of the flame, a volume equal to unity, which exacts the condition that in taking account of the volume of the solid products, the volume of the envelope should be $1 + a$.

We will call this quantity the *force of the powder.*

In a preceding work, supposing with several other authors that the products of combustion of powder, and more generally of any explosive substance, were totally converted into gases, at the temperature of the explosion, we have called the force of the explosive substance the

pressure developed by the unit of weight of the substance burning in the unit of volume.

This definition accords with the preceding, since, under this hypothesis, the volume α of the solid residue becomes zero; but it will be seen that the new definition is more general.

4. The results of the English experiments accord, in fact, very well with formula (3). MM. Noble and Abel have verified this with the pressures which they obtained by burning pebble powder in a closed vessel, the densities Δ varying by increments of 0.1 from $\Delta = 0.1$ to $\Delta = 1.0$.

The summary of their memoir does not mention the values admitted for the values of the constants f and α. These values seem to have been calculated by the aid of the pressures which corresponded to the values $\Delta = 0.6$ and $\Delta = 1$.

We have calculated these quantities by using in their determination all the results observed, using French units. We have found:

$$\alpha = 0.6833 \qquad (4)$$
$$f = 219300,$$

the units being the *decimeter and the kilogram.*

The following table shows the pressures measured, the pressures calculated by formula (3), and the corresponding differences.

We have divided the pressures by 100, consequently in the table the pressures are in *kilograms per square centimeter.*

Densities. Δ	Pressures. Measured.	Calculated.	Differences.
0.1	231	235	— 4
0.2	513	508	+ 5
0.3	839	828	+ 11
0.4	1173	1207	— 34
0.5	1684	1666	+ 18
0.6	2266	2230	+ 36
0.7	3006	2943	+ 63
0.8	3942	3869	+ 73
0.9	5112	5127	— 15
1.00	6567	6926	—359

The differences are not important, and since they do not seem to follow any law, they may be imputed to errors of experiment; besides, the account before mentioned does not give any detail of the method adopted for the measure of the pressures, and it is impos-

sible to discuss the results with exactness. It is likely that the experiments on the pressure of the gases of the powder in a closed vessel are those that were described by Captain Noble in a lecture before the Royal Society of Great Britain, or, at least, that they were determined in the same manner, that is, by *crusher gauges*, and the readings of these instruments being open to discussion, the results found should be accepted with some reserve. We add that the results differ completely from those which Rumford has deduced from his experiments.

In the meantime, formula (3) represents the facts with an exactness such that it is allowable to consider as very plausible the hypothesis according to which the products of the combustion of the powder will be, *under certain circumstances*, partly solid and partly gaseous. The pressure observed being due to the permanent gases alone, may be calculated by Mariotte's law, taking into account the volume occupied by the solid residue.

We show, in the following chapter, how, under this hypothesis, the equation of the movement of the projectile in the interior of a gun may be established.

5. *Temperature of the products of combustion.*—The value of f being determined by experiment, we deduce from it by equation (2) the value of T_0. We have, in fact:

$$T_0 = \frac{273 \cdot f}{p_0 v_0}.$$

According to MM. Noble and Abel, the value of v_0 was 280 litres for the powder experimented with. By making

$$f = 219300$$
$$v_0 = 280$$
$$p_0 = 103.33$$

we find for the absolute temperature of the gases

$$T_0 = 2070°$$

6. In order to calculate this temperature theoretically, it is necessary to know:

1st. The heat lost by the products of the combustion of the unit of weight of powder without production of exterior work by being lowered from the temperature T_0 of combustion to a determined temperature, say zero centigrade or 273° of absolute temperature.

2d. The mean specific heat of the products of combustion between these limits of temperature.

In fact, designating by Q the heat of combustion, and by c the specific heat, we have the relation:

$$Q = c(T_0 - 273),$$

whence

$$T_0 = 273 + \frac{Q}{c}.$$

The heat of combustion can easily be determined by the burning of powder in a calorimeter. MM. Noble and Abel found it equal to 705 *calories* (French units of heat) for the experimental powder.

With regard to the specific heat, it is unknown. MM. Bunsen and Schischkoff have admitted the value $c = 0.185$ for the products of combustion of a powder similar to our sporting powder. But this quantity corresponds to a temperature nearly the mean atmospheric temperature, and, following MM. Noble and Abel, it should vary, *increase*, probably, with the temperature.

Therefore, in default of more precise data, we admit the value $c = 0.185$ with $Q = 705$, and we find

$$T_0 = 4080°,$$

that is to say, a value nearly double that which has been deduced from the measure of the pressures. One would be led to admit, then, that the mean specific heat has a value nearly double that which MM. Bunsen and Schischkoff have adopted, and, consequently, since the specific heat of the gases under constant volume is independent of the temperature, that the specific heat of the solid residue at the temperature of combustion is more than double what it is at ordinary temperatures.

This result may seem excessive. We shall see later, however, in studying the cooling of the gases by the envelope, that one can explain the difference between the theoretic temperature and that deduced from the measure of the pressures, without necessarily admitting this enormous variation of specific heats.

7. *Volume of the solid residue at the temperature of combustion.* The coefficient α, the value of which is determined (4), represents the volume in cubic decimeters, at T_0, of the solid products of combustion of a kilogram of powder. Also, after MM. Noble and Abel, the volume of these products at ordinary temperature is 0.3, and their weight is 0.57 kilogram.

It is easy to conclude from these numbers and from the value $\alpha = 0.6833$:

1st. That the mean coefficient of cubic expansion of the solid residue between the temperature 273° and 2070° is 0.000624, or $\frac{1}{1600}$ nearly;

2d. That their specific weight at the temperature $T_0 = 2070°$ is equal to 0.903.

This last result does not agree with one determined by MM. Bunsen and Schischkoff. These experimenters have, in fact, found, by a method which seems to be a direct experimental determination, that the specific weight of the solid residue is 1.520 at the temperature of 2808°. It seems difficult, at this stage of our knowledge, to decide which of these two numbers is the nearer the truth.

It is, however, to be remarked, that the value $\frac{1}{1600}$ found for the coefficient of expansion notably exceeds analogous coefficients which correspond, in the ordinary limits of determinations, to the greater number of solid bodies. It may then be possible that this coefficient was really too large. The same may be true of a, the coefficient which served to determine it. We shall see, in fact, that an error of this kind can result from the cooling of the products of combustion by the wall of the vessel enveloping them.

8. We will close this discussion with a remark on the comparative volumes of the powder and the solid residue produced by the combustion. The volume of a kilogram of powder of a density 1.8 is $\frac{1}{1.8} = 0.55$. The volume of the solid residue resulting from its combustion is 0.68, according to MM. Noble and Abel, and 0.44 according to MM. Bunsen and Schischkoff, and the volume of the powder is somewhere between these two values. One may conclude from this, that if the solid residue of the combustion of the powder is formed under the same conditions as those of actual service, it would not be a great error to suppose that the volume of the residue, at the temperature of the flame, is equal to that of the powder itself. In the following chapter the utility of this remark will be manifest.

9. *Effect of the cooling of the products of the combustion of powder by the wall of the envelope.*—It has generally been supposed, in theoretical researches on the effects of powder and other explosive substances, that the cooling of the products of combustion by the wall which enveloped them could nearly always be neglected. But this effect cannot be neglected.

We have referred in our preceding work to the calorimetric experiments by which M. Saint-Robert ascertained that, in firing a gun, the

heat absorbed by the walls of the bore could be taken as nearly one-fourth the heat of the combustion of the charge.

The loss of heat, relatively less in guns of large calibre, is certainly a sensible quantity, and Captain Noble, in a lecture on this subject, is disposed to attribute to this cause considerable losses of work observed in certain experiments.

M. Berthelot also estimates that, in the application of processes which we arrange to produce very high temperatures, a large proportion of the living calorific force is lost through the wall of the envelope. This eminent chemist expresses himself as follows, and adds that the facts which support the statement will perhaps not be without interest for the study of the reactions produced by the combustion of powder in the bores of guns:

"L'existence des hautes températures en principe et la possibilité de les réaliser, me paraissent devoir être distinguées avec soin.

"En principe, nos théories actuelles indiquent qu'une masse gazeuse donnée peut acquérir une force vive indéfiniment croissante, c'est-a-dire une température illimitée . . . . .

"Mais, en fait, il se peut que l'intensité des radiations de toute nature augmentant avec une extrême promptitude à mesure que la température s'élève, et par suite les déperditions de la force vive qui se communique aux milieux environnants devenant de plus en plus considérables, rendent irréalisable toute température qui passerait une limite voisine de 2500 ou 3000 degrés observés dans les expériences de M. Sante-Claire Deville."

10. It seems to us probable that in all the circumstances of the combustions of the powder, the intensity of the thermic state realized is such that, in spite of the brevity of the phenomenon, the temperature can be lowered considerably in an extremely short time, and that perhaps to this rapid lowering of the temperature, varying with the mass of the powder, the surface of the envelope, and the period of combustion, may be attributed the difference, which the results present, of various authors who have sought to measure the pressure of the powder gases in a closed vessel.

We submit, while on this subject, some considerations, which without pretending to a rigorously exact determination, will serve, we believe, to give a notion of what these perturbing influences may be.

11. We consider a weight ϖ of powder burning in a closed vessel. The combustion of the powder is not instantaneous; it is done progressively, so that after a time t counted from the beginning of igni-

tion, the quantity of powder burned is a function $F(t)$ of which the form depends upon the physical qualities of the powder and upon the circumstances of combustion.

During combustion the envelope absorbs heat, and, after the time t, the temperature T of the products of combustion is less than T_0, which would have been the temperature had combustion taken place instantaneously. We will endeavor to express this difference. To this end, denote by

σ the total surface of the envelope,

h the velocity with which the heat is imparted for a unit of surface of the wall of the envelope,

The heat absorbed by the envelope after the time t is: $\sigma \int_0^t h.dt.$

Again, the heat lost by the weight $F(t)$, the temperature of the products of which is lowered from T_0 to T, is $cF(t)(T_0 - T)$, c being the mean specific heat under constant volume.

We have then:

$$cF(t)(T_0 - T) = \sigma \int_0^t h.dt. \tag{5}$$

The coefficient of cooling h is a function of the temperatures of the products and of the envelope, vanishing with their difference.

If one admits that the cooling takes place by radiation from the products of combustion through a superficial layer of the envelope, and that this radiation follows the law of Dulong and Petit, we are led to put

$$h = H\epsilon\epsilon'(a^T - a^{T_1}), \tag{6}$$

designating by,

T the absolute temperature of the products of combustion,

T_1 that of the envelope;

ϵ the emissive power of the products of combustion;

ϵ' the absorbent power of the substance of the envelope;

a is a constant always equal to 1.0077;

H is a constant, common to all bodies, and of which the value is 0.000237, if we take for units the decimeter and the second, the emissive and absorbent powers being referred to those of lampblack. The unit of heat is the *calorie* (the quantity of heat necessary to raise the temperature of a kilogram of water one degree centigrade).

NOTE I.—1. The numerical value adopted for the constant H of formula (6) Chapter I, has been deduced from the results of experiments by Dulong and Petit, in their researches on the laws of cooling. Consider a body at the absolute temperature T, cooling in an enclosure to the temperature T_1. Calling σ its

limiting surface, the quantity of heat which traverses this surface during an infinitely small time dt is $h\sigma dt$, h being the quantity defined by formula (6) before cited.

If then we designate by p the weight of the body, and by c its specific heat, the corresponding fall in its temperature is:

$$dT = \frac{h\sigma}{pc} \cdot dt,$$

it follows that the thermometric velocity of cooling is

$$h' = \frac{h\sigma}{pc}.$$

Or better, replacing h by its value,

$$h' = H\frac{\sigma\varepsilon\varepsilon'}{pc}(a^T - a^{T_1}). \tag{1}$$

2. Suppose, in particular, that the body, the cooling of which we are observing, is a sphere of radius r, Δ being the specific weight of the substance, we have

$$\sigma = 4\pi r^2, \quad p = \frac{4}{3}\pi r^3\Delta, \quad \frac{\sigma}{p} = \frac{3}{r\Delta}.$$

Consequently, putting $T = T_1 + \theta$, the formula (1) becomes,

$$h' = H\frac{3\varepsilon\varepsilon' a^{T_1}}{cr\Delta}(a^\theta - 1). \tag{2}$$

3. This being granted, Dulong and Petit have studied the cooling of a mercurial thermometer in an enclosure coated with lampblack to the temperature of zero centigrade.

The bulb of the thermometer was 6 centimeters in diameter, and one can, neglecting the cooling of the glass, apply the calculation to the cooling of a sphere of mercury of a radius of 3 centimeters, by taking for the value of the superficial emissive power the quantity $\varepsilon = 0.8$, which is very nearly that of glass.

Under these conditions, the numerical value of the elements which enter into formula (2) are, taking for units the decimeter and the kilogram:

$\varepsilon = 0.8$	$T_1 = 273$	$r = 0.3$
$\varepsilon' = 1.$	$c = 0.0333$	$\Delta = 13.596$

Under these conditions the experiment gave for the velocity of cooling per minute: $h' = 2.037\,(a^\theta - 1)$

or per second $h' = 0.03395(a^\theta - 1)$.

Comparing this value with formula (2) we deduce from it the value $H = 0.000237$.

12. In consequence of the absorption of heat, the temperature T_1 of the envelope is raised during combustion, which fact tends to diminish the velocity of loss of heat by the products of combustion. But the law of this heating is entirely unknown, and we are obliged to neglect it, supposing that T_1 remains always less than T.

In consequence of this hypothesis, which gives a superior limit to the loss of heat, relation (6) reduces to the following:

$$h = H\varepsilon\varepsilon' a^{T}, \tag{7}$$

and equation (5) should be:

$$cF(t)(T - T_0) = H\varepsilon\varepsilon'\sigma \int_0^t a^{T}.dt, \tag{8}$$

or

$$zF(t) = k\int_0^t a^{-z}.dt, \tag{9}$$

designating by z the fall $T_0 - T$ of the temperature, and putting for brevity

$$k = \frac{H\varepsilon\varepsilon'\sigma a^{T_0}}{c}. \tag{10}$$

In equation (9), the relation which connects z and t, z can be found in an explicit form when we know $F(t)$.

13. If we suppose, for example, the combustion to be uniform, we have $F(t) = \varpi\frac{\tau}{t}$, ϖ being the weight of the powder and τ the period of its combustion.

Equation (9) is satisfied then by giving to z a value independent of t, and this value is obtained by solving the transcendental equation:

$$\frac{\varpi z}{\tau} = ka^{-z}. \tag{11}$$

For any other law of combustion $F(t)$ can generally be developed following the powers of t in the form $F(t) = \varpi at(1 + \lambda t + \mu t_2 + \ldots)$ We develop z in the same manner by putting:

$$z = z_0 + z_1 t + z_2 t^2 + \ldots$$

and substituting in (8) we easily determine the coefficients. This calculation offers but little interest and we will not enter into details. The ideal case of a uniform combustion suffices also for our beginning, which, we repeat, is an approximate representation of the phenomenon more than an absolute measure, which the want of physical data does not permit us to enter upon.

Equation (11), which determines, in this case, the constant fall of the temperature, is solved easily by trial when we know the value of k. It suffices to put the equation in this form:

$$\log z + z\log a = \log k + \log\tau - \log\varpi, \tag{12}$$

the logarithms are those of the common system. We find quickly, by the aid of tables, the value of z, which makes the value of the first member of the equation sensibly equal to the given value of the second member.

14. It is necessary in order to make the calculation to have a value at least approaching that of k, and to know, consequently, the coefficients ε, ε'. The first is the emissive power of the products of combustion: this element is absolutely unknown. It is known, however, that the flame, containing solid incandescent particles, has a considerable emissive power. In the absence of all precise data, we will make the calculation under the hypothesis that it has an emissive power equal to unity.

The absorbent power ε' depends upon the nature of the envelope. For polished iron we can take Leslie's number, $\varepsilon' = 0.15$. We will take finally for the specific heat of the products the value $c = 0.185$ admitted by MM. Bunsen and Schischkoff, and for T_0 the theoretical value of No. 6, $T_0 = 4080$ degrees.

Putting these values in (9) we find:

$$\log k = 9.874 + \log \sigma,$$

consequently equation (12) becomes

$$\log z + z \log a = 9.874 + \log \sigma + \log \tau - \log \varpi.$$

15. Suppose, in order to fix these ideas, that the combustion takes place in a cubic capacity of one decimeter. The surface σ is then equal to 6 (sq. dec.), and, giving to τ and to ϖ various values, we can calculate the corresponding values of z.

We give, in a table, the fall of temperature obtained in supposing:

$\tau = 0.1$ 0.01 0.001 (seconds)
$\varpi = 0.1$ 0.5 1.0 (kilogram)

Values of τ	Values of ϖ (Kilogram).		
Seconds.	k 0.1 degrees.	k 0.5 degrees.	k 1.0 degrees.
0.001	1630	1440	1360
0.010	1910	1720	1630
0.100	2190	2000	1910

These results show that a fall of temperature in the neighborhood of 2000° is not improbable, and consequently we can conclude that the temperature of explosion is, perhaps, not very far from the theo-

retical value, and that the influence of the envelope alone prevents it from reaching that value. In all cases it is not necessary to admit, to explain the difference, a considerable variation of the specific heats.

16. For the same value of τ, the values of z vary with the weight ϖ of the powder burned; however, the variations should be very small, in order that formula (3) may be applicable, supposing f to be constant. But it is easy to see, then, that one would take a value too great of α.

Suppose, in fact, that it is desired to determine α by the aid of two experiments giving the pressures p_1 and p_2, corresponding to the values Δ_1, Δ_2 of Δ. The temperatures being different in the two cases, we have two different values f_1, f_2 of f. The relation (3) gives the two equations:

$$p_1 = \frac{f_1 \Delta_1}{1 - \alpha \Delta_1}, \quad p_2 = \frac{f_2 \Delta_2}{1 - \alpha \Delta_2}.$$

We find from them

$$\alpha = \frac{\dfrac{p_2}{\Delta_2} - \dfrac{p_1}{\Delta_1}}{p_2 - p_1} - \frac{f_2 - f_1}{p_2 - p_1},$$

an expression in which, f increasing with p, the second term is necessarily negative. The exact value of α is then less than that which we should have found had we supposed $f_2 = f_1$.

This result confirms the remarks which were made in the preceding (Nos. 7 and 8).

CHAPTER II.

General Equation of Motion in the Bore.

17. If we admit that a part of the products of combustion are in a solid state, in the ordinary conditions of the employment of powder in guns, we are led to modify in some respects the analysis by which we have already established the equation of motion upon the supposition that the entire products are in a gaseous state.

We may, in fact, assume two hypotheses, which include the truth probably. We may suppose that the solid products are always in equilibrium of temperature with the gas, and thus give up some heat which is transformed into work; or that the temperature of the solid products is constant during the expansion of the gas.

Following the first hypothesis, generally accepted by Noble and Abel, we have calculated (Part I, Chapter 3) the theoretical maxi-

mum work which the powder can perform. Bunsen and Schischkoff have, on the contrary, adopted the second hypothesis, and have calculated the work from the expansion of the permanent gases alone, supposing that the solid residues preserve always their initial temperature.

18. *First hypothesis.*—Suppose first, as in the preceding memoir (Part II, No. 3), that the heat lost by the totality of the products of combustion represents the external work $\mathfrak{E}$ according to the relation,

$$\frac{\mathfrak{E}}{E} = cy\,(T_0 - T), \tag{13}$$

E being the mechanical equivalent of heat;
c the mean specific heat, in constant volume, of the products of combustion;
$y = F(t)$, their weight at the time t;
T their absolute temperature at any time; and
T_0 the temperature of combustion.

The pressure of the permanent gases is given by,

$$p = \frac{p_0 v_0 y\, T}{273 \cdot V}, \tag{14}$$

V being the volume of the gas, and v_0 the specific volume of the gas of a kilogram of powder. Eliminating T between (13) and (14), we have,

$$pV + 2\theta\,\mathfrak{E} = fy, \tag{15}$$

putting

$$2\theta = \frac{1}{273}\,\frac{p_0 v_0}{Ec}, \tag{16}$$

f being the force of the powder, defined by (2) of the preceding chapter.

This equation is similar to (7) of Part II. It differs from this only in the value of 2θ; which is equal, when all the products are gaseous, to $n-1$, n being the ratio of the two specific heats.

The general equation of motion may be deduced as in Part II (No. 5). It is as follows:

$$(u+z)\frac{d^2u}{dt^2} + \theta\left(\frac{du}{dt}\right)^2 = \frac{fy}{m}. \tag{17}$$

The quantity z is defined by the condition that $\omega\,(u+z)$ represents the volume V occupied by the gas; this volume is composed of the following:

1st. The initial volume V_0 around the charge before the displacement of the projectile.

2d. The volume y_1, being that of the interstices between the grains in the fraction of the charge which is lighted.

3d. The cylindrical space ωu, corresponding to the displacement of the projectile.

When the gasification is complete we must add the volume of the powder burned; but, if there are any solid residues, we must take the excess of the volume of the powder over that of the residue formed by the combustion. If then we suppose these two volumes equal (Chapter I, No. 8), we may neglect this factor, and write

$$V = V_0 + y_1 + \omega u,$$

and, consequently,

$$z = \frac{1}{\omega}(V_0 + y_1).$$

When the inflammation is *instantaneous*, the volume $V_0 + y_1$ is always the same as that of the powder chamber, diminished by the volume of the grains. Consequently, the value of z will be constant.

This value may be put in the two forms,

$$z_0 = u_0\left(1 - \frac{\Delta}{\delta}\right) \tag{18}$$

$$z_0 = \frac{\varpi}{\omega}\left(\frac{1}{\Delta} - \frac{1}{\delta}\right) \tag{19}$$

(See Part II, No. 21), where

u_0 is the reduced length of the powder chamber, or the length of a cylinder of equal volume having ω, the cross section of the bore, as a base;

ϖ the weight of the charge;

ω the section of the bore;

Δ the density of loading;

δ the density of the powder.

We see that here the substitution of z_0 for z, adopted as an approximation in Part II, No. 23, to simplify the integration, becomes exact when we adopt the hypothesis of the solid residues.

19. *Second hypothesis.*—Suppose, secondly, that the gaseous products only change their temperature. Let ε be the weight of the gas of a kilogram of powder.

The weight y of powder burned in the time t gives a quantity εy of gaseous product; which, falling from T_0 to T, gives out a quantity of

heat $c_1 \varepsilon y (T_0 - T)$; c_1 being the specific heat, in constant volume, of the gaseous products of combustion alone. Consequently (13) is replaced by

$$\frac{\mathfrak{G}}{E} = c_1 \varepsilon y (T_0 - T).$$

The rest of the deduction in No. 18 remains the same; and (17) is the same if we write εc_1 in place of c in (16), which thus becomes

$$2\theta = \frac{1}{273} \cdot \frac{p_0 v_0}{E c_1 \varepsilon}. \tag{20}$$

Since v_0 is the reduced volume of a weight ε of permanent gas, the *specific* volume or volume of unit weight of the same gas is $\frac{v_0}{\varepsilon}$, and we have consequently

$$E = \frac{1}{273} \cdot \frac{p_0 v_0}{\varepsilon (c' - c_1)},$$

c' being the specific heat under constant pressure.

Note.—The above expression for E may be obtained by writing $\frac{v_0}{\varepsilon}$ for v_0 in (14) Part I.

If then we call n the ratio of the two specific heats $\frac{c'}{c_1}$, we have

$$\theta = \frac{n-1}{2}, \tag{21}$$

as in the case where all the products are assumed to be gaseous.

20. In the two hypotheses, the equation of motion (17) has the same form, but the coefficient θ has different values. We shall now give these values in some particular cases.

First hypothesis.—The value of θ (16) depends upon the volume v_0 and the specific heat c. The first may be determined by experiment. We have elsewhere made this determination for the powders commonly used in France. The second is not known; we consider, however, that it is very nearly the same for all powders (Part I, p. 17); so that we may, without much error, adopt the value found by Bunsen and Schischkoff, $c = .185$.

The following table gives the values of v_0 and those corresponding of θ, for the principal powders.

Name of Powder.	v_0 (liters).	θ.
Sporting powder,	234	.0549
Cannon "	261	.0612
Fine-grained powder, called B	280	.0657

Second hypothesis.—In the second hypothesis we have $\theta = \frac{n-1}{2}$, n being the ratio of the two specific heats of the permanent gases. If we adopt the value $n = 1.4$, which is correct for the state of perfect gas, we have $\theta = \frac{1}{5}$; this is very much larger than the foregoing values.

21. It appears difficult to decide which of these two hypotheses is nearest the truth. In our first researches we admitted that the sensible heat of the totality of the products of combustion would be transformed into work; and this hypothesis was necessary, because we supposed that all these products were in the state of gas or of vapor. If, as the facts set forth in the preceding chapter seem to show, solid products really exist at the temperature of the flame, it is very unlikely that these products, whose emissive power is probably very considerable, would have always the same temperature. But it is possible that they radiate to the walls of the gun, and that the heat should be thus absorbed without greatly modifying the thermic qualities of the gas, whose emissive and absorbing powers are probably very small.

The second hypothesis does not then seem inadmissible. Noble and Abel reject it formally and consider that "the hypothesis of Bunsen and Schischkoff, that the effect on the projectile must be attributed to the permanent gases, without gain or loss of heat, is incompatible with observed facts."

To establish this incompatibility completely, and exhibit the partial, at least, utilization of the heat of the solid products, it would be necessary, it appears, to show that the effective work of a gun may exceed the mechanical equivalent of the sensible heat of the gases produced by the explosion of the charge. For, if the contrary takes place, we may suppose that the difference represents the effect of passive resistances, coolings, interior motions, or other causes of dissipation of energy.

Now, taking for our calculations the data determined in our experimental researches, we find that the theoretical maximum effect, calculated by the hypothesis of Bunsen and Schischkoff, is sensibly greater than the observed effects. Let us enter in some detail upon this question.

22. We have elsewhere remarked (Part II, No. 7) that the equation of motion (17) becomes directly integrable when the explosion is supposed instantaneous. We deduce from this a formula for velocity

which is not exact, since the combustion is progressive and not instantaneous; but which, as observed by Résal in his *Recherches sur le Mouvement des Projectiles*, cannot be very far from the truth, and should give, in all cases, a superior limit of the attainable velocity.

We may establish this formula in the two hypotheses, and compare the theoretical velocities which result with actual velocities. The following is the detail of the calculation:

Let $v = \frac{du}{dt}$, the velocity of the projectile at any time. The integration of (17) in the case of instantaneous explosion gives the relation

$$v^2 = \frac{f\varpi}{\theta m}\left[1 - \left(\frac{z_0}{u + z_0}\right)^{2\theta}\right]. \tag{22}$$

23. In the first hypothesis, f and θ having the values (2) and (16), we have

$$\frac{f}{\theta} = 2Ec\,T_0,$$

or calling Q the heat of combustion of the powder (Part I, No. 3),

$$\frac{f}{\theta} = 2EQ.$$

Consequently, (21) becomes

$$v^2 = 2EQ\,\frac{\varpi}{m}\left[1 - \left(\frac{z_0}{u + z_0}\right)^{2\theta}\right]. \tag{23}$$

The exponent θ has, according to the powder used, one of the values in the table of No. 20.

24. In the second hypothesis, in consequence of (20), we find,

$$\frac{f}{\theta} = 2\varepsilon c_1 E T_0 = 2\,\frac{\varepsilon c_1}{c}\,EQ,$$

and, taking $\theta = \frac{1}{2}$, we have,

$$v^2 = 2\varepsilon\,\frac{c_1}{c}\,EQ\,\frac{\varpi}{m}\left[1 - \left(\frac{z_0}{u + z_0}\right)^{\frac{1}{2}}\right], \tag{24}$$

where

ε is the weight of the permanent of a kilogram of powder,

c the mean specific heat of all the products,

c_1 the specific heat, in constant volume, of the gaseous products.

For the powders used by Bunsen and Schischkoff, $c_1 = .164$, $c = .185$, $\frac{c_1}{c} = .887$; and we may use the figures approximately for the various usual powders.

25. Let us apply formulæ (23) and (24) to some particular cases. The following table gives the elements and result of the calculations.

Names of Elements of calculation units: decimeter and kilogram.	Small arm.	Cannon of 24.	Navy Cannon of 24 cm.
Diameter of the bore,	.175	1.53	2.42
Initial distance of the base of the projectile from the muzzle, u	9.80	27.91	34.45
Reduced length of the chamber, u_0	"	2.95	7.63
Density of loading, Δ	1.00	.537	.798
Reduced length of initial air-space, z_0	.0435*	1.92†	4.24‡
Weight of powder, ϖ	.0025	2.90	28.
Weight of projectile, mg	.036	24.0	144.
Heat of combustion, Q	849	795	795
Weight of the gas of one kilogram of powder, ε	.337	.412	.412
Velocity by (23) (meters)	494	482	560
" " (24) "	365	447	533
" observed "	275	334	432

Remarks.—The powders used are: 1st, sporting powder for the small arm; 2d, cannon powder (poudre à canon des pilons) for the cannon of 24; 3d, Wetteren powder (grains of 13 to 16 mill.) for the navy gun of 24 cm.

The results shown in this table may be thus summarized:

1st. The velocities calculated by (24) (that is, neglecting the transformation of the heat of the solid residues into work) are, in each case, very largely in excess of the true velocity.

2d. For the two large guns, the velocities calculated by the two formulæ differ little.

3d. For the small arm, where the length of the piece is proportionately greater, the velocity calculated by (23) is much greater than the other, and is more than double the true velocity.

It appears difficult to explain this. A considerable fraction of the heat, it is true, is lost to the walls of the gun, but this cannot exercise

* Calculated by (19).

† Calculated by (18).

‡ The proportions of the Wetteren powder are those of the regulation cannon powder; we have taken the same values of Q and ε for these two powders. The values used are those of our first researches on the force of explosive substances.

so great an influence. In fact, whatever might be the law of the absorption of heat, the effect of the heat lost is less than if that heat were to completely disappear at the origin of motion. We must then replace Q by $Q-Q_1$, and consequently multiply the velocities by $\left(1-\frac{Q_1}{Q}\right)^{\frac{1}{2}}$; which will $=.865$, if we take $\frac{Q_1}{Q}=\frac{1}{4}$, following the experiments of Saint-Robert.

The velocities thus calculated are 428 and 316 meters; and the difference between the first of these and the true velocity, which is 275, is certainly too great to be attributed to the other causes of loss of velocity.

The calculation seems therefore to indicate that the heat of combustion of the charge is only partially utilized. It is possible, and even probable, that the temperature of the solid residues is not greatly lowered; or even that the heat which they give out, if they are cooled, does not sensibly heat the gases, and therefore is not transformed into useful work.

Finally, the hypothesis of Bunsen and Schischkoff, on the mode of utilization of the heat of the powder, seems applicable to the expansion of gas in guns. There is nothing to indicate, moreover, that this hypothesis may also be applied to the other modes of using powder; and, in particular, that it should be applied in calculating the theoretical maximum work of indefinite expansion. We may therefore adhere to the views expressed in Part I, p. 33.

26. *Influence of the proper motion of the products of explosion.*—It was assumed in Part II, No. 2, to establish the general equation of motion, that we may approximately substitute for the real state of the products of explosion, a fictitious state in which the entire mass of gas has the mean density and temperature of its different parts. Starting with this hypothesis, we have calculated the pressure upon the projectile and the external work done, according to the laws of Mariotte and Gay-Lussac, as if the products were in a static state.

We have thus analyzed the principal circumstances of the phenomena, making use only of some elementary notions of thermodynamics, which we have deduced from the laws of permanent gases. This analysis is incomplete, as it took no account of the motion of the products of combustion and of the unburned portion of the charge. We may, in the following manner, take account of this, according to the new theory concerning gases.

27. At any instant, let

y = the weight of the products of explosion.

μ = the mass of the charge.

$d\mu$ = the mass of an element of the mixture formed by the burned and unburned fractions of the charge.

x = the distance, a function of the time, which separates the element $d\mu$ from the bottom of the bore.

T = the mean temperature of the products of combustion.

If the products of combustion are formed without proper sensible motion and without the production of external work, their temperature will always remain equal to T_0, the temperature of combustion. But, in consequence of internal motion, and of the work of expansion, their temperature will fall, and the heat given out by the weight y* of the products, falling from T_0 to T, represents half the living force of the charge and the external work produced.†

Equation (13) should then be replaced by the following,

$$Ecy(T_0 - T) = \mathfrak{C} + \frac{1}{2}\int\left(\frac{dx}{dt}\right)^2 d\mu. \tag{25}$$

Also, relation (14) between the pressure p upon the projectile, and the volume, weight and temperature of the products of combustion, must be modified as follows, to take account of the sensible internal motions:

$$pV = \frac{p_0 v_0}{273} yT - \int x \frac{d^2x}{dt^2} . d\mu. \tag{26}$$

NOTE I.—Equation (26) may be established by the aid of considerations similar to those which Clausius and Yvon-Villarceau have recently employed to establish some important theorems of general mechanics.

Consider a material system formed of the products of combustion, and let $d\mu$ be an element of the system. Let x be its distance at any time from the bottom of the bore, and X the component, parallel to this distance, of the force applied to it. We have

$$\frac{d^2x}{dt^2} . d\mu = X.$$

---

* We here reason upon the hypothesis that the totality of the products of explosion are in equilibrium of temperature. If, on the contrary, we suppose that the temperature of the gaseous products alone is lowered by the expansion, we must, as in No. 19, replace c by εc, ε being the weight of the gas of a kilogram of powder, and c its specific heat.

† We neglect the internal work, which, in perfect gases, is considered to vanish, or to become so small that it may be neglected.

Multiplying by x, and summing the results thus obtained for all the elements of the system, we have

$$\int x \frac{d^2x}{dt^2} \cdot d\mu = \Sigma x X. \tag{1}$$

Neglecting the internal forces, according to the theory of gases, the forces applied to the system are reduced to the reactions at the extremities of the mass; that is, to forces equal and opposite to the pressures upon the bottom of the bore and upon the projectile.

We need take no account of the first, for at that point x is zero. If, then, adopting the notation of No. 28, we call

$u+u_0$ the distance which separates the projectile from the bottom of the bore;

p the pressure, on unit surface, upon the projectile;

ω the cross section of the bore, supposed equal to that of the projectile;

the second member of (1) reduces to $-(u+u_0)\omega p$, that is, to $-Vp$, V being the volume occupied by the products of explosion. We have then

$$pV = -\int x \frac{d^2x}{dt^2} \cdot d\mu. \tag{2}$$

2. We must distinguish, in the second member, between the parts which represent the sensible motion, and those which represent the rapid vibratory motions which, according to present theories, constitute the thermic state of the system. According to these theories, each point oscillates around a mean position, and we have $x = x_0 + a$,

x_0 being the abscissa of the mean position, and a the variable displacement in the vibratory motion. Consequently

$$x \frac{d^2x}{dt^2} = x_0 \frac{d^2x_0}{dt^2} + a \frac{d^2x_0}{dt^2} + x_0 \frac{d^2a}{dt^2} + a \frac{d^2a}{dt^2}. \tag{3}$$

3. Consider the *mean values* of the terms in the second member in an interval of time, which is very great compared to the period of thermic motion, but very small in absolute value, so that the variations of x_0 and $\frac{d^2x_0}{dt^2}$ may be neglected. Observing that the means of a and $\frac{d^2a}{dt^2}$ are zero, we may write (3) in the form,

$$x \frac{d^2x}{dt^2} = x_0 \frac{d^2x_0}{dt^2} + a \frac{d^2a}{dt^2},$$

recollecting that the second member is to be replaced by its mean value.

We have thus,

$$pV = -\int a \frac{d^2a}{dt^2} d\mu - \int x_0 \frac{d^2x_0}{dt^2}.$$

The second term represents the value to which pV is reduced when there is no sensible motion in the medium. Now, according to Gay-Lussac's law,

$$pV = \frac{p_0 v_0}{273} \gamma T.$$

Consequently, in the general case,

$$pV = \frac{p_0 v_0}{273} yT - \int x_0 \frac{d^2 x_0}{dt^2} d\mu.$$

Suppressing the subscripts, which are not necessary when we consider the sensible motions alone, we have (26).

Eliminating yT between (25) and (26), we find the relation,

$$(27) \qquad pV + \int x \frac{d^2 x}{dt^2} d\mu + 2\theta \left(\mathfrak{G} + \tfrac{1}{2} \int \left(\frac{dx}{dt} \right)^2 d\mu \right) = fy,$$

which replaces (14).

28. To deduce from this the equation of motion, we will call $u + u_0$ the distance of the projectile from the bottom of the bore; u_0 being its initial distance and u its displacement.

The distance x of any element is evidently less than $u + u_0$; also, if there are no *tourbillons intérieurs*, the velocity and acceleration of this element are less than those of the projectile; we may therefore write,

$$(28) \qquad \int x \frac{d^2 x}{dt^2} d\mu = \alpha \mu (u + u_0) \frac{d^2 u}{dt^2},$$

$$(29) \qquad \int \left(\frac{dx}{dt} \right)^2 d\mu = \alpha' \mu \left(\frac{du}{dt} \right)^2,$$

α and α' being coefficients less than unity.

Also, calling m the mass of the projectile, and z the quantity defined in No. 18,

$$(30) \qquad pV = m(u + z) \frac{d^2 u}{dt^2},$$

$$(31) \qquad \mathfrak{G} = \tfrac{1}{2} m \left(\frac{du}{dt} \right)^2.$$

Finally, putting

$$(32) \qquad \beta = \frac{u + u_0}{u + z},$$

formula (28) may be written,

$$(33) \qquad \int x \frac{d^2 x}{dt^2} d\mu = \alpha \beta \mu (u + z) \frac{d^2 u}{dt^2}.$$

Recollecting the relations (33), (29), (30), (31), (27) becomes

$$(34) \quad (u + z) \frac{d^2 u}{dt^2} \left(1 + \alpha \beta \frac{\mu}{m} \right) + \theta \left(\frac{du}{dt} \right)^2 \left(1 + \alpha' \frac{\mu}{m} \right) = \frac{fy}{m}.$$

Such is the equation of motion of the projectile. It differs from the one already obtained only in the coefficients of the first member.

These coefficients are reduced very nearly to unity when the mass of the charge is very small compared to that of the projectile.

This condition is not realized in the ordinary circumstances of practice; the ratio $\frac{p}{m}$ is often equal, and sometimes superior, to $\frac{1}{3}$. The coefficients may, therefore, differ sensibly from unity.

To ascertain their value, it would be necessary to know the law by which the velocities and accelerations of the various elements of the charge vary. This was attempted by Lagrange, Poisson, and Piobert. The problem is too complex and difficult, however, for us to hope to attain results which would be practically useful.

Fortunately, the importance of its solution is only secondary, for it can be shown, that without making a complete calculation, (17), which was obtained by neglecting the internal motions, takes account of the most essential parts of the problem.

29. We remark first that, in the ordinary conditions of practice, the coefficients α and α' have smaller values than they would have if the entire charge was instantly gasified at the origin of motion.

At any given instant, the solid residues of the burned powder and the grains not yet burned are irregularly dispersed through the mass of gas, and are surrounded by it, so that the pressure on their surfaces is everywhere about the same. Their accelerations and velocities must, consequently, be small, and the corresponding elements of the integrals (28) and (29) will have an insensible value; these are therefore diminished.

Consider now their values if all the charge became gas at the origin of motion. Suppose that an element of the mass, perpendicular to the bore, had at all its parts the same velocity and acceleration parallel to the axis of the gun, without having other motion. Let ρ be the density of the element and dx its thickness. Its mass is $\rho\omega dx$, ω being the cross section of the bore; consequently (28) and (29) become

$$\omega\int x\,\frac{d^2x}{dt^2}\cdot\rho dx \text{ and } \omega\int\left(\frac{dx}{dt}\right)^2\cdot\rho dx.$$

30. To accomplish the integration, it is necessary to know the law by which the density, acceleration, and velocity of an element depend upon its position.

Piobert assumed that its velocity is proportional to its distance from the bottom of the bore. We have then,

$$\frac{dx}{dt}=\frac{x}{u+u_0}\cdot\frac{du}{dt}. \tag{35}$$

Differentiating totally, with respect to t, we have,

$$\frac{d^2x}{dt^2}=\frac{1}{u+u_0}\frac{dx}{dt}\cdot\frac{du}{dt}+\frac{x}{u+u_0}\frac{d^2u}{dt^2}-\frac{x}{(u+u_0)^2}\left(\frac{du}{dt}\right)^2,$$

or, replacing $\frac{dx}{dt}$ by its value in (35)

$$\frac{d^2x}{dt^2}=\frac{x}{u+u_0}\frac{d^2u}{dt^2}. \tag{36}$$

The law of accelerations is therefore the same as the law of velocities.

The density ρ varies throughout the mass according to an unknown law. But the variation is probably not great, and we may neglect it in the calculation of terms of small numerical value.

In consequence of these hypotheses, we find,

$$\int x\frac{d^2x}{dt^2}\,d\mu=\frac{\rho\omega}{u+u_0}\frac{d^2u}{dt^2}\int x^2dx,$$
$$\int\left(\frac{dx}{dt}\right)^2d\mu=\frac{\rho\omega}{(u+u_0)^2}\left(\frac{du}{dt}\right)^2\int x^2dx.$$

The integral $\int x^2dx$ must be taken between the limits 0 and $u+u_0$; its value is $\frac{1}{3}(u+u_0)^3$. Substituting in the two expressions above, and recollecting that $\rho\omega(u+u_0)=\mu$, we have finally

$$\int x\frac{d^2x}{dt^2}\,d\mu=\frac{1}{3}\mu(u+u_0)\frac{d^2u}{dt^2},$$
$$\int\left(\frac{dx}{dt}\right)^2d\mu=\frac{1}{3}\mu\left(\frac{du}{dt}\right)^2;$$

and, consequently, comparing with (28) and (29),

$$\alpha=\alpha'=\frac{1}{3}.$$

These values may be considered a superior limit. A half at most of the charge becoming gas probably, their real values do not perhaps exceed $\frac{1}{4}$.

31. When the gasification is complete, β (32) becomes equal to unity. When the gasification is partial and progressive, we may also take $\beta=1$. In fact, β approaches the value unity for increasing values of u. When u is very small, β differs from unity; but then, the mass of the gaseous products being very small, α and α' are very small and the value of these terms becomes small.

If in (34) we put $\alpha=\alpha'=\frac{1}{3}$, $\beta=1$, we have

$$(u+z)\frac{d^2u}{dt^2}+\theta\left(\frac{du}{dt}\right)^2=\frac{fy}{m\left(1+\frac{1}{3}\frac{\mu}{m}\right)}. \tag{37}$$

This is essentially of the same form as (17), there being a slight change in the second member. We may consider this change as affecting the mass of the projectile, which, if we take $\frac{\mu}{m} = \frac{1}{5}$, should be increased by $\frac{1}{15}$th. If we take, (65) of Part II, the relative variation of the term depending upon m is $\frac{1}{15}$; it would be $\frac{1}{70}$ if we take $a = a' = \frac{1}{4}$, in accordance with the remark at the end of No. 30. In this last case, the variation of the velocity is only 5 or 6 meters.

32. The disturbances which result from the movement of the various parts of the charge appear then to be of small importance; we may then neglect them, and consider that (17) expresses the motion of the projectile correctly. It is also to be noticed that the effect of these motions being to diminish the value of the second member of this equation, has the same effect as diminishing the value of f, and this diminution may be considered a constant quantity, the ratio $\frac{\mu}{m}$ varying very little in the conditions of practice.

We may also, by a change in the value of f, take account of the cooling effect upon the gas of the walls of the gun; the effect of this is probably larger than that which we have just examined. Supposing then that f represents the force of the powder thus corrected, (17) becomes exact.

In this case, the datum *à priori* of the force of the powder appears nearly useless, as we do not know the correction which should be applied to take account of the various causes of loss of work. But we should not attach too much importance to this circumstance. It appears indeed very difficult, and we may add of no practical utility, to arrive at a theory which would enable us to compute any desired result, without certain data determined by experimentation in the same conditions. A theory which requires a small number of ballistic experiments for the determination of a single coefficient, and which will then enable us to calculate effects under other conditions, will, on the other hand, suffice for all practical purposes. The most important point is to establish the *form* of the relations which connects the obtained effects with the variables, and thus to avoid the use of empirical formulæ which may be incompatible with the nature of the laws they are intended to represent.

CHAPTER III.

Combustion of Powder under Variable Pressure.

33. In a previous paper,* was established the general form of the function by which we are able to represent the combustion of a charge of powder. The form of this function depends upon the particular manner in which the combustion is effected, in the different cases; the progessive ignition of the charge and the combustion of the grain. We shall only consider here the particular case in which the conditions of loading are such that the period of ignition may be neglected with respect to that of combustion. This case is realized in practice in consequence of the adoption of large-grained powder for guns of large calibre.

The combustion of the charge is then represented by the same function as the combustion of each grain. If we designate by ϖ the weight of the charge, $\psi(i)$ the fraction of one of the grains (supposed equal) constituting the charge which is burned after a time t, counting from the instant when the surface of the grain is reached by the flame, we have, for the weight of powder burned after a time t: $y = \varpi\psi(t)$.

34. In order to determine, for the various forms of grains, the function $\psi(t)$, we have admitted, with Piobert:

1st. That all the points of the surface of the grain are reached simultaneously by the flame.

2d. That the combustion of the material is propagated, *with a constant velocity*, normal to the surfaces in ignition.

The first of these hypotheses can be accepted without giving rise to sensible error; but the second is certainly not exact. We have heretofore stated† that, zero with the pressure, the velocity of combustion is a rapidly increasing function of the pressure of the medium in which the combustion takes place.

We have thus far neglected this circumstance in the calculation of velocities. In order to comprehend that the results thus obtained can be but little removed from the truth, it suffices to recall the following method which shows the development of useful work in a gun.

Starting from the beginning of ignition the interior pressure rapidly rises, and attains a maximum when the projectile is displaced a very small fraction of the length of the bore. It falls subsequently by

* Part II, chap. 2. † Part II, 17.

amounts which, for progressive powders, are little different the one from the other for the greater part of the path of the projectile. It is during this last period of the phenomenon that the principal useful effect is developed. It is conceivable, then, that one can, with a certain approximation, attribute to the velocity of combustion a constant mean value, by taking, it being well understood, for this element a value different from that which is observed under the normal atmospheric pressure.

In these new researches we propose to study the initial movement; the influence of the pressure on the combustion is then dominating, and cannot be dispensed with.

35. We find in the "Memoires Scientifiques" of M. de Saint-Robert, the complete summary of the experiments which have been made on the velocity of combustion of powder under pressures equal to or less than one atmosphere. But the extent of these experiments is too limited in order to take from them useful data for interior ballistics. In a remarkable *Étude sur les poudres du nouveau matériel de l'artillerie de terre*,* M. Castan shows some facts of experiment for pressures greater than one atmosphere, and remarks that the influence of the pressures on the combustion of powder is made manifest by the examination of firings under the same conditions.

If we admit that the charge of a gun is completely burned when the projectile leaves the bore, one ought to conclude, from the time taken by the projectile to move through the bore, that the period of combustion of powder is much less than that observed in free air. M. Castan estimates, for example, that a powder of which the velocity of combustion is 10 millimetres a second in free air, should burn with a mean velocity of 320 millimetres under the variable pressure produced in large-calibred guns for naval use.†

* Revue d'artillerie, Vol. 1, p. 105.

† It is often useful to have the *mean pressure* produced in a gun, that is to say, the *constant* pressure necessary to produce an observed initial velocity. It is calculated as follows:

Let v = the initial velocity,

m = the mass of the projectile,

u = the space passed over by the projectile in the bore.

The constant force corresponding to the velocity v is evidently $\frac{1}{2} \cdot \frac{mv^2}{u}$; dividing by the area of a right section of the bore ω, we have for the mean pressure

$$P_m = \frac{1}{2} \cdot \frac{mv^2}{\omega u}.$$

36. We have no accurate data on the form of the relation which connects the velocity of combustion of powder with the exterior pressure. In the course of these researches we suppose that the velocity of combustion is proportional to a positive power of the pressure.

According to this hypothesis, which is besides the most simple that one can make in order to represent analytically a function increasing with the variable and vanishing with it, if we designate by:

v_0 the velocity of combustion under the normal atmospheric pressure p_0,

v the velocity under the pressure p,

α a positive number;

we have:

$$v = v_0 \left(\frac{p}{p_0}\right)^{\alpha}. \quad (38)$$

The experiments made up to this date (1877) are not sufficient to determine α with precision. In the meantime we can conclude with certainty that the velocity increases less rapidly than the pressure, that is to say, that α is less than unity.

The following considerations, which, however, we do not state as exact, lead to the value $\alpha = \frac{1}{2}$.

37. The movement of the inflaming gases penetrating into the interior of the substance of the powder can be considered as placed under the action of two opposing forces, which are the exterior pressure p, and the resistance of the material, which, arising from the movement of the gases, is a function $f(v)$ of their velocity.

These two forces are in equilibrium, since the movement of the gases is uniform under constant pressure, and we have the relation,

$$p = f(v).$$

This being granted, the resistance $f(v)$, reduced to the unit of mass of the gases, is probably proportional:

1st. To the velocity v, that is to say, to the number of particles of the resisting medium which are opposed in the unit of time to the movement of the gases.

2d. To a certain function of the velocity vanishing for $v = 0$, and proportionally representing the resistance due to each particle of the medium.

In admitting that this last function should be proportional to the velocity, the function $f(v)$ would be proportional to the square of the velocity, and it follows from the relation $p = f(v)$, the velocity would be *proportional to the square root of the pressure.*

38. Before examining the effect of the pressure on the laws of the combustion of powder, it is well to recall briefly the divers forms which one should attribute to the function $\psi(t)$ when the combustion takes place under constant pressure. This function is generally developed according to integral powers of the variable, and in our first researches we have supposed it reduced to the form

$$\psi(t)=at(1+\lambda t+\mu t^2+\dots),$$

$a, \lambda, \mu \dots$ being coefficients depending on the form and dimensions of the grain.

In the researches which follow we will modify a little this notation, by putting

$$\psi(t)=\frac{at}{\tau}\left(1-\lambda\frac{t}{\tau}+\mu\frac{t^2}{\tau^2}+\dots\right), \tag{39}$$

τ being the period of the combustion of the grain, and $a, \lambda, \mu, \dots$ (*), coefficients which, in consequence of the new form adopted, are purely *numerical*, and retain the same value when the grain varies or remains similar to itself, the period of time τ being alone modified according to the ratio of similitude. We give hereafter the values of these coefficients for various forms of grains.

39. 1st. *Spherical grains.*—When the grain is spherical, designating by r its radius and by v the velocity of combustion, we have the well-known formula,

$$\psi(t)=1-\left(1-\frac{vt}{r}\right)^3,$$

or better, observing that $r=v\tau$:

$$\psi(t)=\frac{3t}{\tau}-\frac{3t^2}{\tau^2}+\frac{t^3}{\tau^3}.$$

Consequently in this case:

$$a=3,\ \lambda=1,\ \mu=\tfrac{1}{3}. \tag{40}$$

40. 2d. *Pierced cylindrical grains.*—This form is not used; the corresponding formula however presents a theoretical interest, because it offers a simplified representation of the law according to which the combustion of pierced prismatic grains operates, these being the forms for some time used in guns of large calibre.

* We write $-\lambda$ in place of λ in the second term of ψ, because this term is negative for the various forms of grains used in practice (No. 39 *et seq.*)

Supposing the grain to be a cylinder of height h and radius r, pierced through its axis by a cylindrical hole of radius r', we have without difficulty the formula

$$\psi(t)=2v\left(\frac{1}{h}+\frac{1}{r-r'}\right)t-\frac{4v^2}{h(r-r')}t^2.$$

The period of combustion is evidently obtained by dividing by v the half of the smaller of the dimensions h and $r-r'$. Denoting by x the ratio of the smaller to the greater of these dimensions we can write:

$$\psi(t)=(1+x)\frac{t}{\tau}-x\frac{t^2}{\tau^2}.$$

From this expression we have the values

$$(41)\qquad a=1+x,\quad \lambda=\frac{x}{1+x},\quad \mu=0.$$

41. 3d. *Grains of the form of a parallelopipedon.*—A very ingenious theory of Captain Castan has recently (1877) called attention to the special properties which may result from the use of grains having very nearly the form of a right parallelopipedon of rectangular base.

Designating by α, β, γ, the dimensions of a grain, the law of its combustion under constant pressure is represented by the formula:

$$\psi(t)=1-\left(1-\frac{2vt}{\alpha}\right)\left(1-\frac{2vt}{\beta}\right)\left(1-\frac{2vt}{\gamma}\right),$$

or better, by developing:

$$\psi(t)=2v\left(\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}\right)t-4v^2\left(\frac{1}{\beta\gamma}+\frac{1}{\gamma\alpha}+\frac{1}{\alpha\beta}\right)t^2+8v^3\frac{1}{\alpha\beta\gamma}t^3.$$

Letting α be the least of the three dimensions, and x, y the ratios of α to β, γ, the period τ of the combustion is evidently determined by the relation $\tau=\frac{\alpha}{2v}$, and the expression above can be written:

$$\psi(t)=(1+x+y)\frac{t}{\tau}-(x+y+xy)\frac{t^2}{\tau^2}+xy\frac{t^3}{\tau^3},$$

and we have consequently:

$$(42)\qquad a=1+x+y,\ \lambda=\frac{x+y+xy}{1+x+y},\ \mu=\frac{xy}{1+x+y}.$$

When the grain is cubical, these values reduce to the values (40) or those of a spherical grain.

42. *Combustion under variable pressure.*—Suppose now a grain of powder to burn under variable pressure. The flame attaining at the

same time every point of its surface, is propagated by parallel surfaces with a variable velocity v; consequently the distance which separates, at any instant t, the surface in ignition and the initial surface is $\int_0^t v dt$ in place of vt as in the case of constant pressure. We have, for example, a spherical grain radius r. The initial volume is $\frac{4}{3}\pi r^3$; at any given instant t the radius has become $r - \int_0^t v dt$, and consequently the volume of powder burned is:

$$\frac{4}{3}\pi r^3 - \frac{4}{3}\pi\left(r - \int_0^t v dt\right)^3;$$

the function $\psi(t)$, the ratio of the actual volume to the initial volume, is then:

$$\psi(t) = 1 - \left(1 - \frac{1}{r}\int_0^t v dt\right)^3.$$

If we admit between the pressure and the velocity of combustion the relation (38), it becomes:

$$\psi(t) = 1 - \left[1 - \frac{v_0}{r}\int_0^t \left(\frac{p}{p_0}\right)^\alpha dt\right]^3,$$

since $\frac{v_0}{r}$ is the period of combustion under the *normal pressure* p_0; supposing then that τ denotes this particular period, the formula above can be written:

$$\psi(t) = 1 - \left[1 - \frac{1}{\tau}\int_0^t \left(\frac{p}{p_0}\right)^\alpha dt\right]^3.$$

It differs only from that relating to the combustion under constant pressure by the substitution of the integral $\int_0^t \left(\frac{p}{p_0}\right)^\alpha dt$ for t.

This result is general and may be extended to other forms of grains. If then the combustion of a grain under the normal pressure p_0 is represented by the function (39), the combustion of the same grain under variable pressure is represented by the formula:

$$(43) \qquad \psi(t) = \frac{a}{\tau}\int_0^t \left(\frac{p}{p_0}\right)^\alpha dt\left[1 - \frac{\lambda}{\tau}\int_0^t \left(\frac{p}{p_0}\right)^\alpha dt + \ldots\right],$$

and the law of the combustion of a weight ϖ of powder formed of equal grains and ignited simultaneously will be: $y = \varpi\psi(t)$.

43. We apply formula (43) to the combustion of the charge in the interior of the gun.

The force applied to the projectile is $m\frac{d^2u}{dt^2}$; dividing by the area of a right section of the bore, we have the pressure for a unit of surface as follows:

$$p = \frac{m}{\omega}\cdot\frac{d^2u}{dt^2}.$$

Substituting in (43), we have for the weight y of the powder burned in the time t:

$$(44)\quad y=\frac{a\varpi}{\tau}\left(\frac{m}{\omega p_0}\right)^a\int_0^t\left(\frac{d^2u}{dt^2}\right)^a dt\left[1-\frac{\lambda}{\tau}\left(\frac{m}{\omega p_0}\right)^a\int_0^t\left(\frac{d^2u}{dt^2}\right)^a dt+\ldots\right].$$

Putting this value in (17) we have the definite equation of the movement of the projectile. We examine in the following chapter the properties and general laws derived from it.

44. *Calculation of the period of combustion.*—In order to calculate the value of τ which enters into these formulæ, it is necessary to know

1st. The velocity of combustion under the atmospheric pressure.

2d. The least dimension of the grain.

The first of these elements has been determined experimentally by Piobert for powders of different composition and fabrication. The result of his experiments is that, all other things being equal, the velocity of combustion of a powder *varies inversely with its density*. We have, therefore,

$$(45)\qquad v_0=\frac{c}{\delta},$$

δ being the density of the grain, and c a constant depending upon the composition and degree of dryness of the material. It depends also upon the process of fabrication and on the period or duration of trituration, but the variations resulting would seem to be relatively small. We can with a sufficiently general approximation suppose that, within the limits of dryness realized in the habitual conditions of fabrication, the value of c to be nearly:

0.200 for powders for war purposes and for English powders;

0.130 for sporting powder (the units being the decimeter and the second).

45. With regard to the least dimension of the grain, it is deduced, generally *from the density* and from the *number of grains in a kilogram.*

1st. *Spherical grains.*—Let a spherical grain have a radius r and density δ, and N be the number of grains to the kilogram. The weight of a grain is $\frac{4}{3}\pi r^3\delta$, and we have: $\frac{4}{3}\pi r^3\delta N=1$, whence

$$(46)\qquad r=\left(\frac{3}{4\pi\delta N}\right)^{\frac{1}{3}}.$$

This form of grain is rarely used in practice, but when the form and the dimensions of the grains of a charge are *very irregular*, the

grain can be assimilated to a *mean* sphere, the radius of which will be given by formula (46).

2d. *Grains of the form of a parallelopipedon.*—Suppose now a grain the form of a parallelopipedon, and let α, β, γ, be the three dimensions, α being the smallest. Put as in No. 41 $\frac{\alpha}{\beta} = x$, $\frac{\alpha}{\gamma} = y$. We have evidently:

$$\alpha\beta\gamma\delta N = \frac{\alpha^3}{xy}\delta N = 1,$$

whence

$$\alpha = \left(\frac{xy}{\delta N}\right)^{\frac{1}{3}}, \tag{47}$$

and when the grain is cubical:

$$\alpha = \left(\frac{1}{\delta N}\right)^{\frac{1}{3}}. \tag{48}$$

These elements being calculated, we have the duration of combustion by the formula $\tau = \frac{r}{v}$, when the grain is spherical or irregular; and by the formula $\tau = \frac{a}{2v}$, when the grain is cubical or of the form of a parallelopipedon.

CHAPTER IV.

General Law and Properties of the Motion of a Projectile in the Bore.

46. We have established in the preceding chapters the general form of the equation which gives the motion of a projectile in the bore. This equation is as follows:

$$(u+z)\frac{d^2u}{dt^2} + \theta\left(\frac{du}{dt}\right)^2 = \frac{fa\varpi}{m\tau}\left(\frac{m}{\omega p_0}\right)^a \int_0^t \left(\frac{d^2u}{dt^2}\right)^a dt \times \left\{ 1 - \frac{\lambda}{\tau}\left(\frac{m}{\omega p_0}\right)^a \int_0^t \left(\frac{d^2u}{dt^2}\right)^a dt + \frac{\mu}{\tau^2}\left(\frac{m}{\omega p_0}\right)^{2a}\left[\int_0^t \left(\frac{d^2u}{dt^2}\right)^a dt\right]^2 + \ldots\ldots\ldots\ldots \right. \tag{49}$$

where

u is the displacement of the projectile,

m its mass;

ϖ the weight of the charge of powder;

ω the right section of the bore;
f the force of the powder;
τ the time of burning of a grain under the normal atmospheric pressure p_0;
$\alpha, \gamma, \mu, \ldots$ numerical coefficients depending upon the form of the grain;
z the reduced length of the initial air space, calculated by (18) or (19);
a the exponent of the power to which we have supposed the velocity of combustion of the powder proportional;
θ the numerical coefficient defined by No. 20.

The problem of the motion of a projectile in a gun is then reduced to finding a function u of t which will satisfy (49), and vanish, with its first derivative $\frac{du}{dt}$, when $t=0$. The numerical determination of this function may be effected by the ordinary methods of approximation used in the integration of differential equations; it requires that the numerical coefficients α and θ shall be known. We shall return to this in the succeeding chapter. But, without this determination, we may deduce from the form of this equation certain general properties of the function u which, from a practical view, appear worthy of remark.

47. Equation (49) depends upon the variables enumerated in the preceding number, and changes with the dimensions of the piece used and the manner of loading. We shall first show that we may substitute for it a system of *purely numerical* equations, and common, therefore, to all cases.

Let y equal the ratio of u to the reduced length of the initial air-space, so that

$$u=yz, \tag{50}$$

and put, for shortness,

$$K=\frac{fa\varpi}{mz^2\tau}\left(\frac{mz}{\omega p^0}\right)^a, \tag{51}$$

$$Y=\int_0^t\left(\frac{d^2y}{dt^2}\right)^a dt; \tag{52}$$

equation (49) then becomes

$$(y+1)\frac{d^2y}{dt^2}+\theta\left(\frac{dy}{dt}\right)^2=KY\left[1-\frac{\lambda}{\tau}\left(\frac{mz}{\omega p_0}\right)^a Y+\frac{\mu}{\tau^2}\left(\frac{mz}{\omega p_0}\right)^{2a}Y^2+\ldots\right]. \tag{53}$$

Change the independent variable t to x, defined by the relation

$$x=K^\beta t, \tag{54}$$

K being the quantity defined by (51), and β an undetermined exponent. We have

$$\frac{dy}{dt} = K^{\beta}\frac{dy}{dx}, \quad \frac{d^2y}{dt^2} = K^{2\beta}\frac{d^2y}{dx^2}. \tag{55}$$

We have also $dt = K^{-\beta}.dx$. From this, and the value of $\frac{d^2y}{dx^2}$ in (55), (52) becomes

$$Y = K^{(2\alpha-1)\beta}\int_0^x \left(\frac{d^2y}{dx^2}\right)^{\alpha}.dx \tag{56}$$

or

$$Y = K^{(2\alpha-1)\beta}X, \tag{57}$$

if we write

$$X = \int_0^x \left(\frac{d^2y}{dx^2}\right)^{\alpha}.dx. \tag{58}$$

Substituting the values from (55) and (57) in (53), and putting,

$$\gamma = 1 + (2\alpha - 3)\beta, \quad \varepsilon = \left(\frac{mz}{\omega p_0}\right)^{\alpha}K^{(2\alpha-1)\beta}, \tag{59}$$

we find,

$$(y+1)\frac{d^2y}{dx^2} + \theta\left(\frac{dy}{dx}\right)^2 = K^{\gamma}X(1 - \lambda\varepsilon X + \mu\varepsilon^2X^2 + \ldots) \tag{60}$$

Determining β so as to make γ vanish, which, by (50) gives

$$\beta = \frac{1}{3-2\alpha}, \tag{61}$$

$$\varepsilon = \frac{1}{\tau}\left(\frac{mz}{\omega p_0}\right)^{\alpha}K^{\frac{2\alpha-1}{3-2\alpha}}, \tag{62}$$

equation (60) is finally reduced to the following,

$$(y+1)\frac{d^2y}{dx^2} + \theta\left(\frac{dy}{dx}\right)^2 = X - \lambda\varepsilon X^2 + \mu\varepsilon^2X^3 + \ldots \tag{63}$$

48. To satisfy this equation, suppose the unknown function y developed in a series of ascending powers of ε, in the form

$$y = y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \ldots, \tag{64}$$

$y_0, y_1, y_2, \ldots$ being unknown functions of x. We have then

$$\frac{d^2y}{dx^2} = \frac{d^2y_0}{dx^2}\left[1 + \varepsilon\frac{d^2y_1}{dx^2}\left(\frac{d^2y_0}{dx^2}\right)^{-1} + \varepsilon^2\frac{d^2y_2}{dx^2}\left(\frac{d^2y_0}{dx^2}\right)^{-1} + \ldots\right]$$

Developing $\left(\frac{d^2y}{dx^2}\right)^{\alpha}$ by the binomial theorem, and putting,

$$(65)\qquad \begin{cases} X_0 = \int_0^x \left(\frac{d^2y_0}{dx^2}\right)^a . dx, \\ X_1 = a\int_0^x \frac{d^2y_1}{dx^2}\left(\frac{d^2y_0}{dx^2}\right)^{a-1} . dx, \\ X_2 = a\int_0^x \frac{d^2y_2}{dx^2}\left(\frac{d^2y_0}{dx^2}\right)^{a-1} . dx \\ \qquad + \frac{a(a-1)}{1.2}\int_0^x \left(\frac{d^2y_1}{dx^2}\right)^2\left(\frac{d^2y_0}{dx^2}\right)^{a-2} . dx \\ \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \end{cases}$$

we easily find for (58),

$$(66)\qquad X = X_0 + \varepsilon X_1 + \varepsilon^2 X_2 + \ldots$$

Substituting the values of y, $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, X, in (63), and equating the coefficients of like powers of ε, we have the following system of equations,

$$(67)\qquad \begin{cases} (y_0+1)\frac{d^2y_0}{dx^2} + \theta\left(\frac{dy_0}{dx}\right)^2 = X_0, \\ (y_0+1)\frac{d^2y_1}{dx^2} + 2\theta\frac{dy_0}{dx}\cdot\frac{dy_1}{dx} + \frac{d^2y_0}{dx^2}y_1 = X_1 - \lambda X_0^2, \\ \left.\begin{array}{l}(y_0+1)\frac{d^2y_2}{dx^2} + 2\theta\frac{dy_0}{dx}\cdot\frac{dy_2}{dx} \\ \quad + \frac{d^2y_0}{dx^2}y_2 + y_1\frac{d^2y_1}{dx^2} + \theta\left(\frac{dy_1}{dx}\right)^2\end{array}\right\} = X_2 - 2\lambda X_0 X_1 + \mu X_0^3, \end{cases}$$

and it will fulfil all the conditions of the problem if we find a system of functions $y_0, y_1, y_2, \ldots$, which will satisfy these equations, and vanish with their first derivatives when $x = 0$.

These equations are numerical, and remain the same whatever may be the conditions of fire, for the same system of values for $\lambda, \mu, \ldots$, that is to say, for the same form of grain.

The functions which are determined by these equations may be tabulated, as in the case of logarithms, and the circular functions; and this labor performed, we can see that (64) furnishes the complete solution of the problem.

49. *General formulæ of initial velocities and pressures.*—The displacement of the projectile u being equal to zy, the velocity v, and acceleration w have the values, from (55)

$$v = zK^\beta \frac{dy}{dx}, \quad w = zK^{2\beta}\frac{d^2y}{dx^2}.$$

To deduce the pressure p on unit surface, we have only to multiply w by m, and divide by ω. If we replace y by its value (64), we have the two formulæ

$$v = zK^{\beta}\left(\frac{dy_0}{dx} + \varepsilon\frac{dy_1}{dx} + \varepsilon^2\frac{dy_2}{dx} + \ldots\right), \tag{68}$$

$$p = \frac{mz}{\omega}K^{2\beta}\left(\frac{d^2y_0}{dx^2} + \varepsilon\frac{d^2y_1}{dx^2} + \varepsilon^2\frac{d^2y_2}{dx^2} + \ldots\right). \tag{69}$$

The coefficients of the second member are numerical coefficients of $x = K^{\beta}t$. These formulæ express then the velocity and pressure as a function of the time. To express them as a function of the distance passed over, we have only to suppose that we have taken from (64) the value of x as a function of y in the form,

$$x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \ldots \tag{70}$$

and substituted this value in (68) and (69), whose coefficients then become functions of $y = \frac{u}{z}$. We have thus

$$v = zK^{\beta}\left[F_0\left(\frac{u}{z}\right) + \varepsilon F_1\left(\frac{u}{z}\right) + \varepsilon^2 F_2\left(\frac{u}{z}\right) + \ldots\right], \tag{71}$$

$$p = \frac{mz}{\omega}K^{2\beta}\left[f_0\left(\frac{u}{z}\right) + \varepsilon f_1\left(\frac{u}{z}\right) + \varepsilon^2 f_2\left(\frac{u}{z}\right) + \ldots\right], \tag{72}$$

which show the forms of the relations which must connect v and p with the various elements of fire. The functions F and f are numerical; the quantities K, ε, β, have the values given in (51), (62), (61).

By giving to u a value equal to the initial distance of the base of the projectile from the muzzle, (71) evidently becomes a formula for initial velocity.

50. *Similar guns.*—Two guns are *similar* when all their homologous linear dimensions are proportional to their calibres. The reduced length of the chamber being then proportional to the calibre, the quantity z, which is itself proportional to the former, varies in the same ratio; the cross section of the bore ω is proportional to the square of the calibre.

The similitude is extended to the loading when the weights of the powder and the projectile are proportional to the cube of the calibre, and when the grains of powder have the same form and dimensions proportional to the calibre. Consequently, the coefficients a, λ, μ, ... must have the same value, and the time of combustion τ must vary, between the two guns, proportionally to the calibre.

When these conditions are fulfilled the relations (51) and (62) show that K^{β} varies in inverse ratio of the calibre, and that ε is constant. It results, therefore, from (71) and (72) that the values of v and p are equal for equal values of $\frac{u}{z}$, that is, for values of u proportional to the calibre.

This principle follows:

In similar guns, similarly loaded, the velocities and pressures corresponding to distances passed over proportional to the calibres, are equal.

This principle requires that the duration of combustion shall be proportional to the calibre; it ceases to be applicable if, all other circumstances of similitude being observed, the powder is the *same* in the two guns. The equality of the velocities and pressures then no longer exists, contrary to the opinion which is very generally held. Consequently, the empirical formulæ, established upon the hypothesis of this equality, are not exact; we shall see further on that the error committed by their use, even in guns which are nearly alike, may be quite large.

51. The principle of the similitude of guns being of great importance in practical applications, it has been thought better to deduce them from the fundamental formulæ in their most general form. In the researches which follow, we shall simplify the formulæ by limiting the number of terms considered. The quantity ε being generally quite small, we may admit that the series (71) and (72) are very convergent, and that the first two terms give a sufficient approximation, similarly to what we have already stated in our first researches, in deriving the expression which gives the initial velocity when the effect of the variation of pressure upon the combustion of the powder is neglected.

The formulæ are thus greatly simplified, and we may, moreover, introduce, in explicit form, the coefficient λ, which, as well as a, depends upon the form of the grain.

52. Consider the development (64) which expresses the displacement of the projectile as a function of the time. The first term, y_0, is defined by the first of (67); which, as has already been remarked, is purely numerical, and does not depend on either λ or μ . . . The coefficient y_1 of the second term satisfies the second of (67). Replacing X_1 in this equation by its value in (65), we easily see that all the terms containing the unknown quantity are *linear*. Moreover, the term

which is independent of the unknown quantity, being proportional to λ, it is clear that y_1 is the product of λ by a numerical function of x.

The second term of (64), and consequently the second term of (70), are then proportional to λ. We have the same results also evidently for the developments (71) and (72), whose second terms are equal to the product of λ by functions of $\frac{u}{z}$.

These functions are, moreover, positive. In fact, if the combustion of a grain under constant pressure were uniform, the coefficients $\lambda, \mu, \ldots$, would vanish; and the series (71) and (72) would be reduced rigidly to their first terms. But the combustion is not uniform in possible shapes of grain; on account of the progressive diminution of the surfaces of emission, the gas is formed with a decreasing velocity; this causes a diminution of the initial velocity and of the rate of increase of pressure in the space passed over. We may then put,

$$F_1\left(\frac{u}{z}\right) = -\lambda\varphi_1\left(\frac{u}{z}\right), \quad f_1\left(\frac{u}{z}\right) = -\lambda\psi_1\left(\frac{u}{z}\right),$$

φ_1, ψ_1 being increasing functions of $\frac{u}{z}$. If, to make the notation symmetrical, we write φ_0 and ψ_0 in place of F_0 and f_0, formulæ (71) and (72), reduced to their first terms, become,

$$v = zK^{\beta}\left[\varphi_0\left(\frac{u}{z}\right) - \varepsilon\lambda\varphi_1\left(\frac{u}{z}\right)\right], \tag{73}$$

$$p = \frac{mz}{\omega}K^{2\beta}\left[\psi_0\left(\frac{u}{z}\right) - \varepsilon\lambda\psi_1\left(\frac{u}{z}\right)\right], \tag{74}$$

whence we derive several important consequences.

53. *Formula of pressures.*—Consider first (74), which gives the pressure.

The maximum pressure, which is what endangers the gun, is generally when the shot has moved a small distance only; the value of p may at that point be reduced to that of its first term. We have then

$$p = \frac{mz}{\omega}K^{2\beta}\psi\left(\frac{u}{z}\right). \tag{75}$$

The function ψ is purely numerical, and entirely independent of the elements of fire. Let b be the value of the variable which corresponds with the maximum, and let A be the value of the maximum. The maximum pressure in the bore is then,

$$P = A\frac{mz}{\omega}K^{2\beta}, \tag{76}$$

or, recollecting the values of K and β in (51) and (61),

$$P = A\frac{mz}{\omega}\left(\frac{fa\varpi}{\tau mz^2}\right)^{\frac{2}{3-2\alpha}}\left(\frac{mz}{\omega p_0}\right)^{\frac{2\alpha}{3-2\alpha}}. \tag{77}$$

Expressed in atmospheres, the pressure is

$$\frac{P}{p_0} = A\left(\frac{mz}{\omega p_0}\right)^{\frac{3}{3-2\alpha}}\left(\frac{fa\varpi}{\tau mz^2}\right)^{\frac{2}{3-2\alpha}}. \tag{78}$$

54. We have called b the value of the variable for which $\phi\left(\frac{u}{z}\right)$ is a maximum. The corresponding value of the displacement is

$$u = bz.$$

We therefore see that, in the degree of approximation adopted, the displacement of the projectile corresponding to the maximum pressure is proportional to the *reduced length of the initial air space.* The only elements of fire upon which its value depends are then, by (18), the reduced length of the chamber, the density of loading, and the density of the powder.

This depends upon the supposition that the expression for the pressure is reduced to its first term. It ceases to be the case if the second term may not be neglected. This may become the case from the use of rapidly burning powders; the value of ε (62) increases as τ decreases.

The value of $y = \frac{u}{z}$, corresponding to the maximum of p, may be obtained by putting the first derivative of $\psi_0(y) - \varepsilon\lambda\psi_1(y)$ equal to zero. We thus have,

$$\psi'_0(y) - \varepsilon\psi'_1(y) = 0. \tag{79}$$

Let $b + \delta b$ be the root we wish to find; we have then

$$\psi'_0(b + \delta b) - \varepsilon\lambda\psi'_1(b + \delta b) = 0.$$

Considering δb a quantity of the first order, noting that $\psi'(b) = 0$, and neglecting the quantities of the second order, we have,

$$\delta b = \varepsilon\lambda\frac{\psi'_1(b)}{\psi'_0(b)}; \tag{80}$$

$\psi''_0(b)$ is negative because $\psi'_0(b)$ is a maximum, $\psi'_1(b)$ is positive because ψ_1 is an increasing function; therefore δb is negative. Consequently the displacement corresponding to the maximum pressure is less than bz, which is therefore a superior limit.

The maximum pressure is a little less than that given by (76). In fact, for the value $b+\delta b$ of the variable, the function $\psi_0(y)-\varepsilon\lambda\psi_1(y)$ is reduced, by neglecting the small quantities of the second order, to

$$\psi_0(b)-\varepsilon\lambda\psi_1(b).$$

This value is less than $\psi_0(b)=A$. In fact, the influence of the second term seems generally small, and (76) and (77) give the pressure with a sufficient approximation for practical purposes, as we shall see later.

55. *Formula for velocity.*—On the contrary, the two first terms must be retained for the initial velocity, which corresponds nearly always to a value of $\frac{u}{z}$ decidedly larger than the one corresponding to the maximum pressure. Upon the ratio between these two terms depends the influence of the form of grain upon the relative values of the velocities and pressures. We shall treat this matter more at length. The formula for velocity, reduced to two terms, when K' and ε are replaced by their values, becomes

$$(81)\qquad v=z\left(\frac{fa\varpi}{\tau mz^3}\right)^{\frac{1}{3-2a}}\left(\frac{mz}{\omega p_0}\right)^{\frac{a}{3-2a}}\left[\varphi_0\left(\frac{u}{z}\right)-\frac{\lambda}{\tau}\left(\frac{fa\varpi}{\tau mz^3}\right)^{\frac{2a-1}{3-2a}}\left(\frac{mz}{\omega p_0}\right)^{\frac{2a}{3-2a}}\varphi_1\left(\frac{u}{z}\right)\right].$$

This is somewhat complicated; and it will be simplified greatly by taking (see No. 37) $a=\frac{1}{2}$. For the moment, however, we shall leave it in its general form, in order to deduce some consequences without putting any unnecessary restrictions upon it.

56. *Influence of the nature of the powder upon its effects.*

1st. *Pressures.*—All other elements being the same, the maximum pressure depends upon the nature of the powder; by (77) this element enters in the quantity $\frac{fa}{\tau}$; which depends upon the three variables f, a, and τ.

f is the force of the powder. If we suppose it given by (2), it is evident that it is proportional to the *temperature of combustion of the powder*, and to the *volume of permanent gas* generated. We may then consider that its value depends principally upon the composition of the powder used.

a is a *number* which depends only on the form of the grain. Its value has been given in No. 39, and the following numbers for the ordinary powders.

τ is the time of combustion of a grain of powder under normal atmospheric pressure. It may be found by dividing the least dimension of the grain by the velocity of combustion; the latter varies, as is known, according to the manner of fabrication.

It results then that if these three elements vary in such a way as to keep $\frac{fa}{\tau}$ constant, the maximum pressure will be constant.

57. 2d. *Initial velocities.*—Equation (81) shows that the initial velocity depends not only upon $\frac{fa}{\tau}$, but also upon $\frac{\lambda}{\tau}$, λ being a second *number* which, like a, is characteristic of the form of the grain.

We see also that the formula for velocity consists of two terms. The first alone would remain in the ideal case where the form of the grain was such as to make the combustion *uniform* under constant pressure. The second term, proportional to $\frac{\lambda}{\tau}$, is subtractive, and represents the effect of the decrease of the velocity of emission which is caused in all possible forms of grain.

It is evidently advantageous to make this term as small as possible, and our formulæ show clearly that, if several powders produce the same maximum pressure under the same conditions of fire, that for which the quantity $\frac{\lambda}{\tau}$ is least gives the greatest velocity.

58. Consider, for example, two powders having the characteristic elements,

$$f, \tau, a, \lambda.$$
$$f', \tau', a', \lambda'.$$

If we suppose that

$$\frac{fa}{\tau} = \frac{f'a'}{\tau'}, \tag{82}$$

the maximum pressure is the same and the ratio of the subtractive terms is $\frac{\lambda'\tau}{\lambda\tau'}$. If this ratio is < 1, the second powder will give a greater velocity with the same pressure.

59. *Influence of the force of the powder.*—Suppose that the form of the grain is the same for two powders, so that $a' = a$, $\lambda' = \lambda$. The equality of pressures is realized if

$$\frac{f}{\tau} = \frac{f'}{\tau'},$$

and the ratio of the two subtractive terms is then $\frac{f}{f'}$. Consequently, the strongest powder gives the same maximum pressure and a greater velocity.

It seems impossible to vary the force of powders of the nitrate of potassium class of powders by varying the proportions. When they are varied, the heat of combustion and the volume of gas vary considerably; but experiment shows that the product of these two quantities, which is an approximate measure of the force, varies very little.

We obtain much more force by using the picrates; which, with equal weights, evolve more heat and more gas than ordinary powder.

This essential condition was probably not realized in certain circumstances in which picric powder was observed to exert a dangerous energy. The foregoing considerations indicate that they may be less destructive than ordinary powder, since, as they can give a greater velocity with the same pressure, they can probably give a lower pressure with the same velocity.

60. *Influence of the form and dimensions of the grain.*—Consider, secondly, two powders which are equally strong: but which differ in the form and dimensions of the grain. Putting $f=f'$, the condition of equality of pressures becomes

$$\frac{a}{\tau}=\frac{a'}{\tau'},$$

and the ratio of the subtractive terms of the velocity is

$$\frac{\lambda' a}{\lambda a'}. \tag{83}$$

For the spherical or cubical grain, we have (Nos. 39 and 41),

$$a=3,\ \lambda=1.$$

Let us compare this grain with other possible forms.

1st. *Cylindrical pierced grain.*—This grain has two dimensions: height and thickness. Calling x the ratio of the least to the greatest, we have (No. 40),

$$a'=1+x,\quad \lambda'=\frac{x}{1+x}.$$

The ratio (83) has then the value

$$\frac{3x}{(1+x)^2}. \tag{84}$$

This function of x is a *maximum* for $x=1$; its value then being $\frac{3}{4}$. Consequently, by using *cylindrical pierced grains of equal height*

and thickness, the subtractive term corresponding to spherical or cubical grains may be diminished by *one-fourth* of its value. If its absolute value is 100 meters, for example, we would gain 25 meters of velocity without increase of maximum pressure. The gain increases for decreasing values of x; if we take $x=\frac{1}{2}$, the subtractive term is diminished by *one-third*; and, supposing the same absolute value of the velocity, the increase is about 33 meters.

2d. *Grains of the form of a parallelopiped.*—Calling x and y the ratio of the least side to the two others, we have (No. 42),

$$a=1+x+y, \quad \lambda=\frac{x+y+xy}{1+x+y}, \tag{85}$$

and the ratio of the subtractive term to that for the spherical or cubical grain is

$$\frac{3(x+y+xy)}{(1+x+y)^2}. \tag{86}$$

If the grain has a square base, we have $y=x$; and the above ratio becomes

$$\frac{3(2x+x^2)}{(1+2x)^2}. \tag{87}$$

This expression diminishes as x varies from 1 to 0, but the change is somewhat slow. When two dimensions of the grain are double the third, $x=\frac{1}{2}$, and the value of the ratio is $\frac{15}{16}$. The velocity therefore increases by $\frac{1}{16}$ of the subtractive term corresponding to the spherical grain. Supposing this term 100 meters, it is about 7; it becomes 16 for $x=\frac{1}{3}$.

The advantage of flattened grains over those which are spherical or cubical seems then real; but the advantage appears to be less than in the case of grains which are pierced.

61. *Theoretical force of some powders.*—The numerical application of the preceding theories requires that the *velocity of combustion* and *force* of the powder shall be known.

The velocity of combustion must be observed under the normal atmospheric pressure, .760m. It has been determined for a great many powders by Piobert, and the figures he has given may still give a sufficient approximation in many cases. In No. 45, some experimental results, which will give at least an approximate value of this element in the ordinary conditions of practice, are given.

The processes of fabrication having, however, been greatly altered since the time of Piobert, it has seemed advisable to undertake new experiments with the new powders.

The *force* of the powder is the element whose direct determination offers the greatest difficulties; there is no experiment which gives its value with certainty for any powder.

In the want of more precise data, we present the *theoretical* values which we derived with M. Roux from the experimental determinations made at the Dépôt Centrale des Poudres et Salpêtres. These values were deduced, by (3) of the first chapter, from the temperatures of combustion and the volume of the permanent gas.

Kind of powder.	Proportions.			Temp. of combustion. T_0 degr.	Volume of gas. v_0 liters.	Force of the powder. f
	Nitre.	Sulphur.	Charcoal.			
Fine sporting powder,	78	10	12	4654	234	412000
Cannon powder,	75	12.5	12.5	4360	261	431000
Small-arm powder, called B,	74	10.5	15.5	4231	280	448000
Commercial powder,	72	13	15	4042	281	430000
Ordinary blasting powder,	62	20	18	3372	307	392000

The units taken are the kilogram and decimeter. Consequently, according to the definition of the force of a powder, the figures in the table express in kilograms the pressure per square decimeter of the permanent gases of a kilogram of powder, occupying, at the temperature of combustion, the volume of one liter.

The figures of this table confirm what was said in No. 59 concerning the equality of the force of powders which have been differently made.

62. If according to No. 37, we take the velocity of combustion of powder to be proportional to the square root of the pressure, we put $\alpha = \frac{1}{2}$ in the preceding formulæ. We thus obtain new and very simple expressions which, as we shall see later, appear to represent accurately facts and the results of experiment.

63. *Formula for pressures, when* $\alpha = \frac{1}{2}$.—Replacing α by $\frac{1}{2}$, and z by its value in (19), equation (77), for the pressure, becomes,

$$P = A \frac{fa}{\tau\omega} \left[\frac{m\varpi\delta\Delta}{p_0(\delta - \Delta)} \right]^{\frac{1}{2}}, \tag{88}$$

where

f is the force of the powder,

τ the time of burning of a grain under the pressure p_0,

ω the right section of the bore,

m the mass of the projectile,

ϖ the weight of the powder,
Δ the density of loading,
δ the density of the powder,
a a numerical coefficient depending upon the form of grain,
A an absolute number whose value is independent of the elements of fire, and of the units chosen.

64. *Determination of A by experiment.*—To determine A theoretically, it would be necessary to integrate the first of (66), and deduce a table of values of the function ψ_0, of which A is the maximum. This determination, which, however, will be effected later, is not necessary. The essential form of the function which gives the maximum pressure being fixed, we have only, to determine A, to measure the pressure in known conditions of fire—a single experiment is enough.

The following is an example of this determination by means of a navy gun of 24 centimeters; Wetteren powder was used (13 to 16 millimeters). The values of the various quantities on which the pressure depended were as follows (kilogram, decimeter and second being the units taken):

$$f = 431000^{*} \qquad a = 3^{\dagger} \qquad \tau = 0.6^{\ddagger}$$
$$m = \frac{144}{g}^{\S}, \qquad \varpi = 26, \qquad \Delta = .762,$$
$$\delta = 1.80, \qquad p_0 = 103.33 \qquad \omega = 4.60^{\P}$$

In these conditions, the mean of three *manomètres à écrasement* in the powder chamber gave 2300 kilos per square cm. We have, therefore, $P = 230000$; from which we have the value

$$A = 0.703.$$

It would be of interest to determine this coefficient by means of a more accurate instrument. This might be done by the use of the instrument lately invented by MM. Déprez and Sébert, and Captain Ricq.

65. *Pressures produced by the same powder in similar guns.*—In passing from one similar gun to another, m and ϖ vary as the cube of

* This value is the one which, in table in No. 62, belongs to cannon powder, whose elements are in the same proportion as in the Wetteren powder.

† The grain being irregular, it was assumed that the value of a for a sphere could be taken.

‡ This value is obtained by taking a velocity of combustion of 11.7 mm. per second, and calculating the mean radius of grain by (46), with the values $N = 350$, $\delta = 1.8$.

§ $g = 98.09$ d.

¶ The calibre is 2.42 d. c.

the calibre, and ω as its square; the density of loading Δ remains the same. If then the powder is the same for the two arms, τ does not change; and we see, from (88), that the maximum pressure varies proportionally to the calibre. Consequently, we may lay down the following law:

In similar guns, charged with the same powder, the maximum pressure is proportional to the calibre.

This law may often be made of use. For example: experiment has shown that the maximum pressure developed by Wetteren powder (13 to 16 mill. grains) in a 24 cm. gun is about 2300 kilos per square cm.; from which we conclude that the pressure of the same powder in a 14 cm. gun will be $2300 \times \frac{14}{24} = 1340$ kilos per square cm. about.

66. *Formula for velocities in the hypothesis*, $\alpha = \frac{1}{2}$.—Putting $\alpha = \frac{1}{2}$ in (81), we find;

$$(89) \qquad v = \left(\frac{fa\varpi}{\tau m}\right)^{\frac{1}{2}} \left(\frac{mz}{\omega p_0}\right)^{\frac{1}{2}} \left[\varphi_0\left(\frac{u}{z}\right) - \frac{\lambda}{\tau}\left(\frac{mz}{\omega p_0}\right)^{\frac{1}{2}} \varphi_1\left(\frac{u}{z}\right)\right].$$

To apply this formula, the functions φ_0 and φ_1 must be known, or (67) must be integrated. But, without this, we may derive from (89) an *approximate* formula for initial velocities which may be useful.

67. *Approximate formula for initial velocity.*—The empirical formulæ by which initial velocities have been determined have generally been *monomial*. The form is much the simplest for calculation by logarithms, and it is not incompatible with the form which theory indicates, since any function may always, *within certain limits*, be considered as nearly proportional to a properly chosen power of the variable.

Adopting this method, we see that the quantity,

$$\varphi_0\left(\frac{u}{z}\right) - \frac{\lambda}{\tau}\left(\frac{mz}{\omega p_0}\right)^{\frac{1}{2}} \varphi_1\left(\frac{u}{z}\right),$$

which is an increasing function of $\frac{u}{z}$ and a decreasing one of $\frac{\lambda}{\tau}\left(\frac{mz}{\omega p_0}\right)^{\frac{1}{2}}$, may be considered, *as an approximation*, as proportional to a positive power γ' of the first of these variables, and to a negative power $-\gamma$ of the second. We may then, calling B a numerical factor, replace it by an expression of the form

$$B\left(\frac{\tau}{\lambda}\right)^{\gamma}\left(\frac{\omega p_0}{mz}\right)^{\frac{\gamma}{2}}\left(\frac{u}{z}\right)^{\gamma'}.$$

Consequently (89) becomes

$$(90) \qquad v = B\left(\frac{fa\varpi}{\tau m}\right)^{\frac{1}{2}}\left(\frac{mz}{\omega p_0}\right)^{\frac{1-2\gamma}{4}}\left(\frac{\tau}{\lambda}\right)^{\gamma}\left(\frac{u}{z}\right)^{\gamma'};$$

where γ and γ' may be determined by experiment. We shall return to this point in a special work.

Supposing that γ and γ' are known (90), enables us to evaluate numerically the influence of the various elements of fire upon the effects obtained. It exhibits, in particular, the proper influence of the characteristics of the powder used. This last, it appears, could hardly be obtained by any purely empirical method.

We shall now deduce from (90) some consequences which seem worthy of remark.

68. *Initial velocity attained by the same powder in similar guns.*— In similar guns, u and z vary with the calibre, ω as the square of the calibre, and m and ϖ as its cube. Also, a, τ and λ remain the same; we therefore conclude that *the initial velocity varies as the* $\frac{1-2\gamma}{2}$ *power of the calibre.*

69. *Initial velocities attained by different powders giving the same maximum pressure.*—Consider two powders having the characteristics,

$$f,\ \tau,\ a,\ \lambda,$$
$$f',\ \tau',\ a',\ \lambda'.$$

In the same conditions of fire these powders would give the same maximum pressure if their characteristics satisfy the relation (82),

$$\frac{fa}{\tau} = \frac{f'a'}{\tau'}.$$

This being fulfilled, formula (90) shows that the corresponding initial velocities are connected by the relation,

$$\frac{v'}{v} = \left(\frac{\lambda\tau'}{\lambda'\tau}\right)^{\gamma},$$

or, since

$$\frac{\tau'}{\tau} = \frac{f'a'}{fa},$$

$$(91) \qquad \frac{v'}{v} = \left(\frac{f'a'\lambda}{fa\lambda'}\right)^{\gamma}.$$

We may thus compare various powders, and ascertain what influence the force of the powder and form of the grain exercise.

70. *Maxima pressures produced by different powders giving the same initial velocity.*—Consider, secondly, two powders which, in the

same conditions of fire, give the same initial velocity, and let us compare their pressures.

According to (90) the equality of velocities requires the condition

$$\left(\frac{fa}{\tau}\right)^{\frac{1}{2}}\left(\frac{\tau}{\lambda}\right)^{\gamma}=\left(\frac{f'a'}{\tau'}\right)^{\frac{1}{2}}\left(\frac{\tau'}{\lambda'}\right)^{\gamma},$$

whence

$$\frac{\tau'}{\tau}=\left(\frac{fa}{f'a'}\right)^{\frac{1}{1-2\gamma}}\left(\frac{\lambda'}{\lambda}\right)^{\frac{2\gamma}{1-2\gamma}}. \tag{92}$$

Also, from (88)

$$\frac{P'}{P}=\frac{f'a'\tau'}{fa\tau};$$

consequently, recollecting (92), we have

$$\frac{P'}{P}=\left(\frac{fa\lambda'}{f'a'\lambda}\right)^{\frac{2\gamma}{1-2\gamma}}. \tag{93}$$

71. *Approximate value of* γ.—In order that (91) and (93) shall be of any practical use, we must know the value of γ at least approximately.

Its value may be found from certain experimental data. Experiments made at Gavre with a 24 centimeter gun show that the initial velocities reached with projectiles of different mass, all other elements of fire being constant, vary inversely as the $\frac{4}{10}$ths power of the mass of the projectile.

Admitting this, it follows from (90) that we must have nearly

$$\frac{1-2\gamma}{4}=\frac{1}{10},$$

and consequently,

$$\gamma=\frac{3}{10}.$$

We have then

$$\frac{2\gamma}{1-2\gamma}=\frac{3}{2};$$

and (91) and (93) become

$$\frac{v'}{v}=\left(\frac{f'a'\lambda}{fa\lambda'}\right)^{\frac{3}{10}}, \tag{94}$$

$$\frac{P'}{P}=\left(\frac{fa\lambda'}{f'a'\lambda}\right)^{\frac{3}{2}}. \tag{95}$$

Equation (94) enables us to compare the velocities given by powders having the same maximum pressure; the condition of equality of pressures is expressed by the equation,

$$\frac{\tau}{\tau'}=\frac{fa}{f'a'}. \tag{96}$$

Equation (95) serves to compare the maximum pressures caused by powders giving the same velocity. Putting $\gamma = \frac{3}{10}$ in (92), we see that the characteristics of the powder are then connected by the relation,

$$\frac{\tau}{\tau'} = \left(\frac{fa}{f'a'}\right)^{\frac{5}{6}} \left(\frac{\lambda'}{\lambda}\right)^{\frac{2}{3}}. \tag{97}$$

The various formulæ which precede can only be considered as approximate, owing to the uncertainty in the value of γ. We have found the value $\gamma = \frac{3}{10}$ from the exponent of the variable m in (90); now, it is easy to see that a very slight variation of this exponent will cause the value of γ, and particularly that of $\frac{2\gamma}{1-2\gamma}$, to change considerably.

Notwithstanding this, we believe that the relations furnished by theory may, with advantage, be applied to the study of ballistics, in place of the formulæ hitherto used, which have all been empirical.

72. To give an example, we shall compare the effects of two powders of the same make and differing in form and density of grain.

Take two powders, the grains of one being spheres and those of the other parallelopipeds. In the latter, let the base be a square, and the side of the base double the height. We have:

1st. For the sphere,

$$a = 3, \quad \lambda = 1;$$

2d. For the parallelopiped,

$$a' = 2, \quad \lambda' = \tfrac{8}{5}.$$

We therefore have from (94) and (95),

$$\frac{v'}{v} = \left(\frac{16}{15}\right)^{\frac{5}{10}} = 1.020,$$

$$\frac{P'}{P} = \left(\frac{16}{15}\right)^{\frac{5}{3}} = .908.$$

By the substitution of the parallelopiped for the sphere, we may therefore increase the velocity by $\frac{1}{50}$ about without changing the pressure, or diminish the pressure by about $\frac{1}{10}$ without changing the velocity.

In the first case, the ratio of the times of burning are, from (96),

$$\frac{\tau}{\tau'} = \frac{3}{2};$$

and, in the second case, from (97),

$$\frac{\tau}{\tau'} = \left(\frac{3}{2}\right)^{\frac{5}{6}} \left(\frac{5}{8}\right)^{\frac{2}{3}} = 1.362.$$

From the ratio of the times of combustion, we can deduce the number of grains to a kilogram.

For the spherical powder, let
r be the mean radius of grain,
δ the density of the powder,
v the velocity of combustion,
N the number of grains to the kilogram.

We have from No. 45

$$\tau = \frac{r}{v}, \quad r = \left(\frac{3}{4\pi\delta N}\right)^{\frac{1}{3}}.$$

Similarly for the second powder, calling δ', v', N', the homologous parts to δ, v, N, and a the height of the grain, we have from No. 45,

$$\tau' = \frac{a}{2v'}, \quad a = \left(\frac{1}{4\delta' N'}\right)^{\frac{1}{3}}.$$

The material of the two powders being the same, we have $v' = v$, $\delta' = \delta$, and consequently

$$\frac{\tau}{\tau'} = 2\left(\frac{3N'}{\pi N}\right)^{\frac{1}{3}}.$$

We deduce from this expression, and from the two values of $\frac{\tau}{\tau'}$ above, that,

1st. To obtain the same maximum pressure we must have

$$\frac{N'}{N} = 0.44\ldots;$$

2d. To obtain the same velocity,

$$\frac{N'}{N} = 0.33\ldots$$

73. *Application to a particular case.*—Wetteren powder (13 to 16 mill. grains), fired in the navy 24 cm. gun, in the ordinary conditions of *épreuves de réception*, gives about 437 meters velocity and 2500 kilos per square centimeter pressure.

The grains of this powder are irregular, and the mean number of grains to the kilogram is 350. Suppose that we substitute for it another powder of the same material, but whose grains are parallelopipeds with square bases and heights $\frac{1}{2}$ the side of the square.

It results from what precedes:

1st. That, if the number of grains to the kilogram of the new powder is $350 \times .44 = 154$,

the initial velocity will be

$$437 \times 1.02 = 446 \text{ meters about,}$$

the pressure being the same as that of the Wetteren powder.

2d. That, if the number of grains to the kilogram is

$$350 \times .33 = 115,$$

the maximum pressure will be

$$2500 \times .908 = 2270 \text{ kilos about,}$$

the initial velocity being the same as that given by the Wetteren powder.

These results would be changed if the constants f, δ, v, differed for the two powders. The general formulæ, however, will permit our calculating the effect of these elements.

We shall return to these questions of application, which seem to have a certain importance from a practical standpoint.

CHAPTER V.

Numerical Determination of the Auxiliary Functions.

74. We have seen (No. 47) that the theoretical study of the movement of the projectile in the interior of a gun should be amended by the determination of certain functions purely numerical. These functions, which may be termed *auxiliaries*, are those which we have designated (No. 48) by the notation $y_0, y_1, y_2, \ldots$ We have remarked that they consist of special transcendentals, the complete numerical determination of which would permit the theoretical solution of all the problems relating to the firing of any gun, whatever may be the variable elements of the firing.

The auxiliary functions are defined by equation (67). It does not seem possible, in the actual state of the resources of analysis, to deduce the explicit form of the unknown functions of equations where they are involved not only under the characteristics of differentiation, but also under the signs of integration.

But we can apply to these equations the methods used for the numerical integration of differential equations. The application of these methods leads to long and laborious calculations, but does not present any theoretical difficulty. We will explain it briefly.

75. *Determination of the function* y_0.—We have designated by y_0 that one of the auxiliary functions which forms the first term of the development of the space traversed by the projectile as a function of the time taken to travel this space. It is defined by the first of equations (67), which we reproduce below, replacing X_0 by its value (65)

$$(98) \qquad (y_0+1)\frac{d^2y_0}{dx^2}+\theta\left(\frac{dy_0}{dx}\right)^2=\int_0^x\left(\frac{d^2y_0}{dx^2}\right)^\alpha dx.$$

It should vanish also, as well as its first derivative for the value $x=0$. For small values of x we satisfy the equation (98) and also the initial conditions by putting:

$$(99) \quad \begin{cases} \dfrac{d^2y_0}{dx^2}=ax^m+bx^n+cx^p+\ldots \\ \dfrac{dy_0}{dx}=\dfrac{a}{m+1}\cdot x^{m+1}+\dfrac{b}{n+1}\cdot x^{n+1}+\dfrac{c}{p+1}\cdot x^{p+1}+\ldots \\ y_0=\dfrac{a}{(m+1)(m+2)}\cdot x^{m+2}+\dfrac{b}{(n+1)(n+2)}\cdot x^{n+2} \\ \quad +\dfrac{c}{(p+1)(p+2)}\cdot x^{p+2}+\ldots \end{cases}$$

$a, b, c, \ldots\ m, n, p, \ldots$ to be determined. Substituting the values (99) in equation (98) and identifying, we find for the exponents,

$$(100) \qquad m=\frac{1}{1-\alpha},\quad n=\frac{4-2\alpha}{1-\alpha},\quad p=\frac{7-4\alpha}{1-\alpha}\ldots;$$

and the coefficients $a, b, c, \ldots$ are given by the equations

$$a=\frac{1}{m^m},$$

$$(101) \qquad \frac{a^2}{(m+1)(m+2)}+b+\theta\frac{a^2}{(m+1)^2}=\frac{am}{n}b,$$

$$\frac{ab}{(m+1)(n+2)}+\frac{ab}{(n+1)(n+2)}+c+2\theta\frac{ab}{(m+1)(n+1)} =\frac{am}{p}c+\frac{a(a-1)m}{2p}\cdot\frac{b^2}{a}.$$

The series (99) are very convergent for values of x less than unity. We can also deduce from them the values of y_0 and of its derivatives for $x=1$, but beyond $x=1$ they become rapidly divergent, and it becomes necessary to have recourse to Taylor's Theorem.

76. Suppose the functions $\frac{d^2y_0}{dx^2}$, $\frac{dy_0}{dx}$, y_0 calculated for a given value of x, the variable.

The increments of these functions corresponding to the increase h of the variable are given by the known developments:

$$(102)\quad \begin{cases} \Delta \dfrac{d^2y_0}{dx^2} = h\dfrac{d^3y_0}{dx^3} + \dfrac{h^2}{2}\cdot\dfrac{d^4y_0}{dx^4} + \dfrac{h^3}{6}\cdot\dfrac{d^5y_0}{dx^5} + \dots \\ \Delta \dfrac{dy_0}{dx} = h\dfrac{d^2y_0}{dx^2} + \dfrac{h^2}{2}\cdot\dfrac{d^3y_0}{dx^3} + \dfrac{h^3}{6}\cdot\dfrac{d^4y_0}{dx^4} + \dfrac{h^4}{24}\cdot\dfrac{d^5y_0}{dx^5} + \dots \\ \Delta y_0 = h\dfrac{dy_0}{dx} + \dfrac{h^2}{2}\cdot\dfrac{d^2y_0}{dx^2} + \dfrac{h^3}{6}\cdot\dfrac{d^3y_0}{dx^3} \\ \qquad\qquad + \dfrac{h^4}{24}\cdot\dfrac{d^4y_0}{dx^4} + \dfrac{h^5}{120}\cdot\dfrac{d^5y_0}{dx^5} + \dots \end{cases}$$

In order to make the calculations it is necessary to have the derivatives of y_0 of a superior order to the second. Differentiating several times equation (98) we obtain the system,

$$(103)\quad \begin{cases} (y_0+1)\dfrac{d^3y_0}{dx^3} + (1+2\theta)\dfrac{dy_0}{dx}\cdot\dfrac{d^2y_0}{dx^2} = \left(\dfrac{d^2y_0}{dx^2}\right)^a, \\ (y_0+1)\dfrac{d^4y_0}{dx^4} + 2(1+\theta)\dfrac{dy_0}{dx}\cdot\dfrac{d^3y_0}{dx^3} + (1+2\theta)\left(\dfrac{d^2y_0}{dx^2}\right)^2 \\ \qquad = a\left(\dfrac{d^2y_0}{dx^2}\right)^{a-1}\cdot\dfrac{d^3y_0}{dx^3}, \\ (y_0+1)\dfrac{d^5y_0}{dx^5} + (3+2\theta)\dfrac{dy_0}{dx}\cdot\dfrac{d^4y_0}{dx^4} + (4+6\theta)\dfrac{d^2y_0}{dx^2}\cdot\dfrac{d^3y_0}{dx^3} \\ \qquad = a(a-1)\left(\dfrac{d^2y_0}{dx^2}\right)^{a-2}\left(\dfrac{d^3y_0}{dx^3}\right)^2 + a\left(\dfrac{d^2y_0}{dx^2}\right)^{a-1}\dfrac{d^4y_0}{dx^4}, \\ \dots\dots\dots\dots\dots\dots\dots\dots \end{cases}$$

The first of these equations gives $\dfrac{d^3y_0}{dx^3}$, the second $\dfrac{d^4y_0}{dx^4}$ and so on for functions of derivatives of superior order. We can then calculate all the coefficients of the developments (102).

We deduce the values of y_0, $\dfrac{dy_0}{dx}$, $\dfrac{d^2y_0}{dx^2}$, for the value $x+h$ of the variable; next those corresponding to $x+2h$, and so on, step by step.

We can also, following the degree of convergence of the series (102), supposed to be reduced to the same number of terms, modify the numerical value of the increase h of the variable.

77. Passing to the numerical calculation it is necessary to have given the values of a and θ. We have made the calculation, supposing a to be $\frac{1}{2}$, according to No. 37, and $\theta = \frac{1}{2}$, following the second hypothesis of No. 20, and we have formed the following table, giving the corresponding values of y_0 and of its first and second derivatives.

x	Values of y_0	$\frac{dy_0}{dx}$	$\frac{d^2y_0}{dx^2}$
1.00	0.021	0.082	0.242
1.25	0.050	0.158	0.363
1.50	0.102	0.264	0.485
1.75	0.184	0.399	0.591
2.00	0.303	0.557	0.666
2.25	0.464	0.729	0.704
2.50	0.668	0.906	0.709
2.75	0.917	1.082	0.690
3.00	1.208	1.250	0.657
3.25	1.541	1.409	0.617
3.50	1.912	1.558	0.574
3.75	2.319	1.697	0.533
4.00	2.760	1.825	0.494
4.50	3.731	2.055	0.427
5.00	4.809	2.254	0.372
5.50	5.981	2.429	0.329
6.00	7.235	2.584	0.294
6.50	8.562	2.724	0.265
7.00	9.956	2.850	0.241
8.00	12.925	3.072	0.205
9.00	16.095	3.263	0.179
10.00	19.443	3.421	0.158

78. *Determination of the function* y_1.—The function y_1 furnishes the second term of the development (64). It is defined by the second of the equations (67), in which by replacing X_0 and X_1 by their values, (65) can be written

$$(104)\quad (y_0+1)\frac{d^2y_1}{dx^2}+2\theta\frac{dy_0}{dx}\cdot\frac{dy_1}{dx}+\frac{d^2y_0}{dx^2}y_1 = \alpha\int_0^x\frac{d^2y_1}{dx^2}\left(\frac{d^2y_0}{dx^2}\right)^{\alpha-1}dx-\lambda\left[\int_0^x\left(\frac{d^2y_0}{dx^2}\right)^{\alpha}dx\right]^2.$$

The function y_1 should satisfy this equation and should vanish, as also its first derivative for $x=0$.

In order to determine it numerically we proceed as in the case of y_0.

For small values of x, y_0 and its first and second derivatives are represented by the developments (99). Equation (104) is satisfied then by representing the ratios of y_1 and its derivatives to λ by analogous developments, that is to say, by putting

$$(105)\quad \begin{cases} \frac{1}{\lambda}\cdot\frac{d^2y_1}{dx^2}=a_1x^{m_1}+b_1x^{n_1}+c_1x^{p_1}+\ldots \\ \frac{1}{\lambda}\cdot\frac{dy_1}{dx}=\frac{a_1}{m_1+1}x^{m_1+1}+\frac{b_1}{n_1+1}x^{n_1+1} \\ \qquad\qquad +\frac{c_1}{p_1+1}x^{p_1+1}+\ldots \\ \frac{1}{\lambda}y_1=\frac{a_1}{(m_1+1)(m_1+2)}x^{m_1+2} \\ \qquad\qquad +\frac{b_1}{(n_1+1)(n_1+2)}x^{n_1+2}+\ldots \end{cases}$$

$a_1, b_1, c_1, \ldots$; $m_1, n_1, p_1, \ldots$ to be determined. Substituting the values (99) and (105) in the equation (104) and identifying similar powers of x, we find the following values of the exponents:

$$(106)\qquad m_1=2m,\quad n_1=n+m,\quad p_1=p+m,$$

and recollecting the values (100),

$$(107)\qquad m_1=\frac{2}{1-\alpha},\quad n_1=\frac{5-2\alpha}{1-\alpha},\quad p_1=\frac{8-4\alpha}{1-\alpha}\ldots$$

The system of equations which determines the coefficients $a_1, b_1, c_1, \ldots$ is the following

$$(108)\quad \begin{cases} a_1\left(1-\frac{\alpha a^{\alpha-1}}{m_1}\right)+\frac{a^{2\alpha}}{m^2}=0 \\ Aa_1+b_1\left(1-\frac{\alpha a^{\alpha-1}}{n_1}\right)+\frac{2\alpha b a^{2\alpha-1}}{mn}=0 \\ Ba_1+Cb_1+c_1\left(1-\frac{\alpha a^{\alpha-1}}{p_1}\right)+D=0; \end{cases}$$

putting for brevity:

$$(109)\quad \begin{cases} A=\left[\frac{1}{(m+1)(m+2)}+\frac{2\theta}{(m+1)(m_1+1)}\right. \\ \qquad\left.+\frac{1}{(m_1+1)(m_1+2)}\right]\alpha-\frac{\alpha(\alpha-1)ba^{\alpha-2}}{n}, \\ B=\left[\frac{1}{(n+1)(n+2)}+\frac{2\theta}{(n+1)(m_1+1)}\right. \\ \qquad\left.+\frac{1}{(m_1+1)(m_1+2)}\right]b-\frac{\alpha(\alpha-1)c^{\alpha-2}}{p_1} \\ \qquad-\frac{1}{2}\frac{\alpha(\alpha-1)(\alpha-2)b^2a^{\alpha-3}}{p_1}, \\ C=\left[\frac{1}{(m+1)(m+2)}+\frac{2\theta}{(m+1)(n_1+1)}\right. \\ \qquad\left.+\frac{1}{(n_1+1)(n_1+2)}\right]\alpha-\frac{\alpha(\alpha-1)ba^{\alpha-2}}{p_1}, \\ D=\frac{2\alpha c a^{2\alpha-1}}{mp}+\frac{\alpha(\alpha-1)b^2a^{2\alpha-2}}{mp}+\frac{a^2b^2a^{2\alpha-2}}{n^2}. \end{cases}$$

These formulæ permit the calculations of y_1, $\frac{dy_1}{dx}$, $\frac{d^2y_1}{dx^2}$ for $x=1$ and for inferior values of the variable.

For values of x greater than unity the series (105) becomes rapidly divergent and cannot be employed.

It is necessary then to calculate by formula (102) the increments of increase of y_1, $\frac{dy_1}{dx}$, $\frac{d^2y_1}{dx^2}$, corresponding to the increase h of the variable. This calculation supposes the initial numerical values of derivatives of superior order to be known. They are obtained by taking the successive derivatives of equation (104).

The problem is then theoretically solved, but the numerical calculations being very long we have omitted them. This calculation also offers no real interest after we have fixed with some certainty the numerical values of the coefficients α and β.

We hope to return again to this subject, and in the meantime present here some supplementary remarks on the function y_0, of which we have given the complete numerical calculation.

79. The numerical determination of y_0 permits the establishment of the laws of the initial movement of the projectile, since in the first instance the value of the first term of the series which represents the displacement is preponderating.

It should also be remarked that an error in the value of θ has not then a very sensible influence because of the relatively small value of the second term of equation (98).

The table of No. 77 can then furnish some useful indications of the law which regulates the interior pressure near the maximum, which is produced as we know when the projectile is displaced only a fraction, generally very small, of the length of the bore.

In view of these applications, it is convenient to change the independent variable, and to form a table of the corresponding values of $\frac{d^2y_0}{dx^2}$ and y_0.

We obtain thus the numerical determination of the function $\psi_0\left(\frac{u}{z}\right)$ which, according to formula (75), serves to calculate the pressure p corresponding to the space u passed over by the projectile.

This calculation offers no difficulty. It is effected very simply by constructing, by the aid of the table of No. 77, a curve having for abscissae y_0 and for ordinates the corresponding values of $\frac{d^2y_0}{dx^2}$. We

deduce graphically by this interpolation the figures of the following table:

y_0	$\frac{d^2y_0}{dx^2}$	y_0	$\frac{d^2y_0}{dx^2}$	y_0	$\frac{d^2y_0}{dx^2}$
0.1	0.480	0.6	0.710	1.25	0.651
0.2	0.605	0.7	0.705	1.50	0.621
0.3	0.665	0.8	0.700	1.75	0.590
0.4	0.693	0.9	0.692	2.00	0.563
0.5	0.700	1.0	0.680	2.50	0.513

It is superfluous to prolong this table for greater values of y_0; the influence of the second term which depends on y_1 becomes more and more sensible, and the value of $\frac{d^2y_0}{dx^2}$ ceases to represent the interior pressure with a sufficient approximation.

We recollect that the variable y_0 represents, in the order of approximation adopted, the ratio $\frac{u}{z}$ of the displacement of the projectile to the *reduced length of the initial air space*, which is calculated according to (18) by the formula:

$$z = u_0\left(1 - \frac{\Delta}{\delta}\right),$$

in which we designate by

u_0 the reduced length of the powder chamber;*

Δ the density of loading;

δ the density of the powder.

80. *Maximum pressure, displacement of the projectile.*—We see by an inspection of the curve which serves to establish the preceding table:

1st. That the maximum of $\frac{d^2y_0}{dx^2}$ is equal to 0.710;

2d. That the corresponding value of y_0 is very nearly 0.6.

The first of these results shows a remarkable verification. The maximum value which we find theoretically equal to 0.710 is that of the coefficient A of No. 63. Now in determining this coefficient by the aid of the experiments furnished by the firing of a 24 centimeter gun, we find $A = 0.703$, a number which differs very little from the preceding.

If, then, in the circumstances of firing in question, we admit *a priori* formula (88), attributing to A the theoretical value 0.710, and

* Defined in No. 18.

to the quantities f and τ the particular values of No. 64, the calculation gives for the maximum pressure a value which differs very little from that obtained by the use of crusher gauges.

Again, the theoretical value of f, deduced by formula (2) from the temperature of combustion and from the volume of the permanent gases of the powder, and the value of τ being the duration of combustion of the grain under the normal atmospheric pressure, we conclude that theory permits us to compute the maximum pressure, in this particular case, when we know the following elements of which the determination can be made by simple experiments in the laboratory:

1st. The heat of combustion of the powder;

2d. The volume (reduced to zero and at 0.760 m.) of the permanent gases produced;

3d. Its velocity of combustion in free air.

It would be interesting to ascertain if this agreement of theory and experiment, of which the true determination is always subordinate to the measure of precision of crusher gauges, is maintained under different conditions of firing from those in which we have verified it.

81. We deduce finally from what precedes a very important result. The maximum pressure is produced when the ratio of the displacement of the projectile to the reduced length of the initial air space is equal to 0.6. Consequently denoting by U the displacement corresponding to the maximum, we have very nearly:

$$U = 0.6u_0\left(1 - \frac{\Delta}{\delta}\right). \tag{110}$$

On the other hand, this result should be considered as approximate only (See No. 54). But the approximation is sufficient, we believe, in the greater number of cases.

Example. In a 24 centimeter gun, fired under the ordinary conditions of powder proof, we have (the decimeter being taken for the unit): $u_0 = 7.63$, $\Delta = 0.800$, $\delta = 1.800$,

and consequently: $U = 2.54$.

82. *Case in which the first auxiliary equation is integrable.*—The equation (98) becomes integrable when the exponent a reduces to unity. This case would be realized if the velocity of combustion of the powder was proportional to the exterior pressure. There is reason to suppose that the velocity of combustion increases less rapidly than the pressure, and the case which we show is consequently purely theoretical. It offers, however, some interest because the first auxiliary function being then taken in explicit form, we are able to judge,

with a precision which is not obtained by the method of numerical approximation which we have employed, of the general aspect of this function in this real case.

Making $a=1$, equation (98) reduces to the following:

$$(y_0+1)\frac{d^2y_0}{dx^2}+\theta\left(\frac{dy_0}{dx}\right)^2=\frac{dy_0}{dx},$$

or better, putting $\frac{dy_0}{dx}=v,$

$$(y_0+1)\frac{dv}{dx}+\theta v^2=v.$$

We have also: $$\frac{dv}{dx}=\frac{dv}{dy_0}\cdot\frac{dy_0}{dx}=v\frac{dv}{dy_0}.$$

Substituting and dividing by v we have

$$(y_0+1)\frac{dv}{dy_0}+\theta v=1,$$

whence $$\frac{dv}{\theta v-1}+\frac{dy_0}{y_0+1}=0,$$

and integrating:

$$(\theta v-1)(y_0+1)^\theta=\text{constant}.$$

The constant is determined by the condition $v=0$, for $y_0=0$. It is equal to -1. We have consequently:

$$v=\frac{1}{\theta}\left[1-\frac{1}{(y_0+1)^\theta}\right]. \tag{111}$$

We find from it by quadrature the relation between x and y_0. The second derivative $\frac{d^2y_0}{dx^2}$ is obtained by multiplying the value of v by its derivative with respect to y_0.

$$\frac{d^2y_0}{dx^2}=\frac{1}{\theta}\left[\frac{1}{(y_0+1)^{\theta+1}}-\frac{1}{(y_0+1)^{2\theta+1}}\right], \tag{112}$$

the value of y_0, which gives the maximum, is obtained from the relation:

$$\frac{1}{y_0+1}=\left(\frac{\theta+1}{2\theta+1}\right)^{\frac{1}{\theta}},$$

and the corresponding value of $\frac{d^2y_0}{dx^2}$ is:

$$A=\frac{1}{\theta}\left[\left(\frac{\theta+1}{2\theta+1}\right)^{\frac{\theta+1}{\theta}}-\left(\frac{\theta+1}{2\theta+1}\right)^{\frac{2\theta+1}{\theta}}\right].$$

On the other hand, in this simple case, the function y_1 does not appear obtainable in an explicit form.

PART IV.

PRACTICAL FORMULÆ FOR VELOCITIES AND PRESSURES IN GUNS.

PREFACE.

In a preceding memoir published in 1876 was established the necessary form of the formulæ which represent the initial velocity and the maximum pressure produced in guns.

We shall take account in this Part of the various circumstances which influence the effects obtained, and notably, the variation that the velocity of the combustion of powder presents under variable pressure.

The formulæ indicate, moreover, the law according to which the results of firing depend upon the form of the grain; an element, the influence of which has become preponderating since the employment, very nearly general to-day, of progressive powders.

In the present memoir will be fixed the numerical value of the coefficients in a way to render the formulæ immediately applicable to the calculation of the various conditions of practice. This determination is made the object of the first chapter.

In Chapter II the formulæ will be verified by the comparison of calculated and measured results, and it will be shown that they represent seventy-five very different conditions of firing, in guns which vary in calibre from 320 to 75 millimeters, with a mean difference of velocities less than 3 meters, the velocities varying from 300 to 550 meters.

Chapter III is devoted to the development of some consequences of the theory, and to the examination of some questions of interior ballistics which have not hitherto been studied, except in an empirical manner.

CHAPTER I.

FORMULÆ FOR VELOCITIES AND PRESSURES.

1. *Formula for velocities.*—In a preceding part the general form of the expression for the initial velocity produced in a gun by the combustion of progressive powder has been shown.

This formula was obtained by supposing the velocity of combustion of the powder to be proportional to a positive power of the pressure. In what follows this power is taken equal to $\frac{1}{2}$. This value is indicated also by theoretical considerations (Art. 36, Part III), and leads, as will be seen later, to the formulæ which accord with the facts.

Under this hypothesis, the expression for the velocity is the following [Eq. (89), Part III]:

$$(1) \qquad v=\left(\frac{fa\varpi}{\tau m}\right)^{\frac{1}{2}}\left(\frac{mz}{\omega p_0}\right)^{\frac{1}{4}}\left[\varphi_0\left(\frac{u}{z}\right)-\frac{\lambda}{\tau}\left(\frac{mz}{\omega p_0}\right)^{\frac{1}{2}}\varphi_1\left(\frac{u}{z}\right)\right],$$

designating by

ϖ the weight of the charge of powder,

m the mass of the projectile,

u the length of the path of the projectile (this length is the distance which separates, at the position of loading, the base of the projectile from the face of the muzzle. This is the length, and not the total length of the bore, which directly influences the velocity),

ω the right section of the bore,

z the reduced length of the initial air space. (*Initial air space* is the difference between the volume of the powder chamber and the volume of the powder. The *reduced length* corresponding is measured by the height of a cylinder having for a base the right section of the bore and for a volume that of the *initial air space*),

f the force of the powder. (The force of a powder is the pressure of the gases of 1 kilogram of that powder occupying, at the temperature of combustion, the unit of volume. See Art. 61, Part II),

p_0 the normal atmospheric pressure,

τ the duration of the combustion of a grain of powder under the pressure p_0,

α, λ numerical coefficients depending upon the form of the grain (these values for various forms of grains will be found, Articles 39, 40, 41, Part III);

φ_0, φ_1 are purely numerical functions.

The reduced length of the *initial air space* is calculated by formula

$$(2) \qquad z=\frac{\varpi}{\omega}\left(\frac{1}{\Delta}-\frac{1}{\delta}\right), \text{ [See Part III, (19)]}$$

in which we call:

Δ the density of loading,

δ the real density of the powder.

2. The theoretical determination of the functions φ_0 and φ_1 offers some difficulties which have been shown in Chapter V, Part III. But this determination is not indispensable for the applications which we have in view, and we can, as will be seen, deduce from the relation (1) a formula which suffices for the wants of practice.

3. Whatever may be the analytical expressions for φ_0 and φ_1, these functions are such that they become respectively proportional to the $\frac{1}{4}$ and $\frac{3}{4}$ powers of $\frac{u}{z}$ for infinitely great values of that variable.

This condition results from the analytical discussion of the equations which define these auxiliary functions.

Also we observe that the two terms of expression (1), respectively proportional to $z^{\frac{1}{4}}$ and $z^{\frac{3}{4}}$, remain finite for infinitely small values of z.

So that we have, when $\frac{u}{z}$ is very great,

$$(3) \qquad \varphi_0\left(\frac{u}{z}\right)=A_0\left(\frac{u}{z}\right)^{\frac{1}{4}}, \quad \varphi_1\left(\frac{u}{z}\right)=A_1\left(\frac{u}{z}\right)^{\frac{3}{4}},$$

A_0 and A_1 representing numerical coefficients.

4. This being granted, we write formula (1) as follows:

$$(4) \quad v=\left(\frac{fa\varpi}{\tau m}\right)^{\frac{1}{2}}\left(\frac{mz}{\omega p_0}\right)^{\frac{1}{4}}\varphi_0\left(\frac{u}{z}\right)\left[1-\frac{\lambda}{\tau}\left(\frac{mz}{\omega p_0}\right)^{\frac{1}{2}}\frac{\varphi_1\left(\frac{u}{z}\right)}{\varphi_0\left(\frac{u}{z}\right)}\right].$$

In the ordinary conditions of practice, the ratio $\frac{u}{z}$ has a large value; on the other hand, the second term of the quantity between brackets is generally a fraction much smaller than unity. We can, therefore, substitute approximately for the ratio $\frac{\varphi_1}{\varphi_0}$ the value which it approaches for increasing values of $\frac{u}{z}$. From equation (3) we have then very nearly

$$\frac{\varphi_1\left(\frac{u}{z}\right)}{\varphi_0\left(\frac{u}{z}\right)}=Q\left(\frac{u}{z}\right)^{\frac{1}{2}}.$$

Admitting this hypothesis, and supposing, further, that the function φ_0 is sensibly proportional within certain limits to a power γ of the variable, that is to say, putting:

$$\varphi_0\left(\frac{u}{z}\right)=P\left(\frac{u}{z}\right)^{\gamma},$$

it becomes:

$$v = P\left(\frac{fa\varpi}{\tau m}\right)^{\frac{1}{2}}\left(\frac{mz}{\omega p_0}\right)^{\frac{1}{4}}\left(\frac{u}{z}\right)^{\gamma}\left[1 - Q\frac{\lambda}{\tau}\left(\frac{mu}{\omega p_0}\right)^{\frac{1}{2}}\right],$$

or better still, replacing z by its value (2),

$$(5)\quad v = Pp_0^{-\frac{1}{4}}\left(\frac{fa}{\tau}\right)^{\frac{1}{2}}\left(\frac{\varpi}{m}\right)^{\frac{1}{4}}\left(\frac{\varpi}{\omega}\right)^{\frac{1}{2}-\gamma}\left(\frac{1}{\Delta} - \frac{1}{\delta}\right)^{\frac{1}{4}-\gamma}u^{\gamma}\left[1 - Q\frac{\tau}{\lambda}\left(\frac{mu}{\omega p_0}\right)^{\frac{1}{2}}\right].$$

5. Designating by p the weight of the projectile, c the calibre or diameter of the bore, g the acceleration of the force of gravity, we have:

$$m = \frac{p}{g}, \quad \omega = \frac{\pi c^2}{4}.$$

Substituting these values in expression (5) and putting for brevity:

$$A = P\left(\frac{g}{p_0}\right)^{\frac{1}{4}}\left(\frac{4}{\pi}\right)^{\frac{1}{2}-\gamma}, \quad B = Q\left(\frac{4}{\pi g p_0}\right)^{\frac{1}{2}},$$

it becomes:

$$(6)\quad v = A\left(\frac{fa}{\tau}\right)^{\frac{1}{2}}\left(\frac{\varpi}{p}\right)^{\frac{1}{4}}\left(\frac{\varpi}{c^2}\right)^{\frac{1}{2}-\gamma}\left(\frac{1}{\Delta} - \frac{1}{\delta}\right)^{\frac{1}{4}-\gamma}u^{\gamma}\left[1 - B\frac{\lambda}{\tau}\frac{(pu)^{\frac{1}{2}}}{c}\right].$$

6. In order to obtain a formula which shall be numerically applicable, it remains to fix the value of γ. It can be obtained from the results of experiments. In fact, the empirical formulæ of the commission of Gavre established the fact that, all other elements being constant, the initial velocity is:

1st. Proportional to the $\frac{6}{10}$ power of the charge (Memorial de l'artillerie de la Marine, Vol. IV, pp. 25 and 45).

2d. Inversely proportional to the $\frac{1}{4}$ power of the capacity S' in which the powder charge is placed (Mem. de l'art. de la Mar., Vol. II, p. 37).

The velocity v is then approximately proportional to

$$\frac{\varpi^{\frac{6}{10}}}{S'^{\frac{1}{4}}} = \varpi^{\frac{6}{10}-\frac{1}{4}}\Delta^{\frac{1}{4}}.$$

Now according to formula (6), the velocity v is proportional to

$$\varpi^{\frac{1}{2}-\gamma}\left(\frac{1}{\Delta} - \frac{1}{\delta}\right)^{\frac{1}{4}-\gamma}.$$

Comparing these two results, we conclude,

1st. That we should have very nearly,

$$\frac{3}{4} - \gamma = \frac{6}{10} - \frac{1}{4},$$

hence

$$\gamma = \frac{4}{10};$$

2d. That the quantity:

$$\left(\frac{1}{\Delta} - \frac{1}{\delta}\right)^{\frac{1}{4}-\gamma} = \left(\frac{\delta\Delta}{\delta - \Delta}\right)^{\frac{3}{2\sigma}}$$

should be, for values of δ and Δ, comprised between the limits of practice, sensibly proportional to $\Delta^{\frac{1}{4}}$, and this is also verified numerically.

We are then led to the formula:

$$(7) \qquad v = A\left(\frac{fa}{\tau}\right)^{\frac{1}{2}}\left(\frac{\varpi}{p}\right)^{\frac{1}{4}}\left(\frac{\varpi}{c^2}\right)^{\frac{1}{10}}\left(\frac{\delta\Delta}{\delta - \Delta}\right)^{\frac{3}{2\sigma}} u^{\frac{4}{10}}\left[1 - B\frac{\lambda}{\tau}\cdot\frac{(pu)^{\frac{1}{2}}}{c}\right],$$

which we can in practical applications replace by the following:

$$(8) \qquad v = A\left(\frac{fa}{\tau}\right)^{\frac{1}{2}}\left(\frac{\varpi}{p}\right)^{\frac{1}{4}}\left(\frac{\varpi}{c^2}\right)^{\frac{1}{10}}\Delta^{\frac{1}{4}}u^{\frac{4}{10}}\left[1 - B\frac{\lambda}{\tau}\frac{(pu)^{\frac{1}{2}}}{c}\right].$$

This last equation is more simple and sufficiently exact. We shall adopt it in our last researches. (Formula (8) does not take into account, as in formula (7), the variations of δ; but these variations are so small that in practice their influence on the velocity may be neglected.)

7. *Characteristics of a powder.*—Formula (8) includes two quantities $\left(\frac{fa}{\tau}\right)^{\frac{1}{2}}$ and $\frac{\lambda}{\tau}$, whose values depend upon the nature of the powder and upon the form and dimensions of the grains. These two elements completely determine the effect produced by a powder under given conditions. We can, then, call them *characteristics*. We will designate them hereafter by the letters α, β, putting:

$$(9) \qquad \alpha = \left(\frac{fa}{\tau}\right)^{\frac{1}{2}}, \quad \beta = \frac{\lambda}{\tau}.$$

8. *Numerical determination of the constants A and B.*—Supposing the characteristics of a given powder to be known, it suffices to measure the velocities given by this powder under two different conditions of firing, in order to find the numerical values of A and B from formula (8). This amounts to replacing successively the various elements in the formula by the particular values relating to the two firings, giving to v the corresponding values determined by experiment, and we have two equations showing the relation between the two unknown quantities A and B.

9. In the absence of precise data on the absolute values of α and β, we may take them equal to 1 for a well-defined powder, and refer the characteristics of other powders to those of the powder adopted as a *standard*. From this point of view we take, for example, as a standard a sample of Wetteren powder (grains of 13 to 16 millimeters), which under the normal conditions of proof firing has given in the 24 and 14 centimeter guns initial velocities respectively of 4371 and 4384 decimeters.

The numerical values of the various elements of firing are shown in the following table:

Name of Gun.	c decim.	u decim.	p kil.	ϖ kil.	Δ	v decim.
24 cent.	2.42	38.19	144	28	0.789	4371
14 cent.	1.40	27.04	21	4	0.891	4384

Putting successively the particular values from this table in formula (8) and replacing $\frac{fa}{\tau}$ and $\frac{\lambda}{\tau}$ by 1, we obtain two equations from which we find the values

$$A = 2088, \quad B = 0.01089. \tag{10}$$

Consequently for any powder whatever of characteristics α and β the formula for velocities is the following:

$$v = 2088\alpha\left(\frac{\varpi}{p}\right)^{\frac{1}{4}}\left(\frac{\varpi}{c^3}\right)^{\frac{1}{16}}\Delta^{\frac{1}{4}}u^{\frac{4}{16}}\left[1 - 0.01089\beta\frac{(pu)^{\frac{1}{2}}}{c}\right], \tag{11}$$

units: *decimeter, kilogram, second.*

10. In place of taking α and β equal to unity for the standard powder, we may calculate the numerical values of these quantities according to certain particular values of f, τ, a, and λ. These last quantities are it is true imperfectly known, but no error will result in the calculation of velocities, since we only change the units to which the characteristics of a powder are referred.

This method of working has the advantage of taking into account in the valuation of characteristics the physical signification of the elements on which they depend. It permits also, in certain cases, the establishment *à priori* and independently of all experiment, the values of the characteristics due to the physical properties of the powder. We shall give hereafter some examples.

In order to determine the constants A and B under this hypothesis, we adopt for the standard powder the values:

$$f = 431000, \; \tau = 0.730, \; a = 3, \; \lambda = 1. \tag{12}$$

[The standard powder is composed as follows: saltpetre 75 parts, charcoal 12.5 parts, and sulphur 12.5 parts. Its specific constants are the following:

Number of grains to the kilogram, ... $N = 350$
Density in mercury, ... $\delta = 1.75$
Velocity of combustion under normal pressure (decimeters) $w = 0.10$

The grain is very irregular, but the mean form is supposed to be a sphere of radius r, and given by the formula

$$r = \left(\frac{3}{4\pi\delta N}\right)^{\frac{1}{3}}.$$

The duration of combustion is deduced from the equation: $\tau = \frac{r}{w}$.

The value $f = 431000$ is that adopted for cannon powder (No. 61, Part III). The values $\alpha = 3$, $\lambda = 1$ are for spherical grains (No. 39, Part III)].

By equation (9) the characteristics of this powder are deduced,

$$\alpha_0 = 1331, \quad \beta_0 = 1.370, \tag{13}$$

and the new constants A and B are found by dividing equation (10) by α_0, β_0, we have thus:

$$A = 1.569, \quad B = 0.00795, \tag{14}$$

consequently the formula for velocities as a function of all the elements of firing becomes

$$v = 1.579\alpha \left(\frac{\varpi}{p}\right)^{\frac{1}{4}} \left(\frac{\varpi}{c^2}\right)^{\frac{1}{16}} \Delta^{\frac{1}{4}} u^{\frac{1}{16}} \left[1 - 0.00795\beta \frac{(pu)^{\frac{1}{2}}}{c}\right]. \tag{15}$$

This formula constitutes the definite result we had in view. We give in succeeding chapters numerous applications of this formula.

The signification of the letters on which it depends has been given in Articles 2, 5, and 7. It is essential to recollect that the units adopted are the decimeter, the kilogram, and the second. (In the original formula (5) the constants P and Q are absolute numbers, independent of the choice of units. In formula (8) however the values of the constants A and B defined by the relations of No. 5 depend upon the units cited above.)

11. *Formula of pressures.*—Supposing the velocity of combustion proportional to the square root of the pressure under which it takes place, the maximum of the mean interior pressure is given by the formula:

$$P = A \frac{fa}{\tau\omega} \left[\frac{m\varpi\delta\Delta}{p_0(\delta - \Delta)}\right]^{\frac{1}{2}}, \tag{16}$$

in which A denotes a numerical factor independent of the choice of units, the signification of the other letters being the same as for formula (15). (See Part III, Nos. 63 and 64. In Part III, Art. 64, the value of A deduced from experiment is given as 0.703, admitting for the Wetteren powder $\tau = 0.6$. This number corresponds to a velocity of combustion of 11.7 millimeters per second. Some results recently obtained, however, lead us to admit that this velocity is in reality nearly 10 millimeters. This value which has served to establish the constants of formula (15) will be adopted in the calculations of pressures.)

Replacing, as in No. 5, m and ω by their values as functions of p and c, designating by K the product of A and all the constant factors of the formula, and observing finally that $\frac{fa}{\tau}$ is the square of the characteristic a defined in No. 7, it becomes

$$P = Ka^2 \frac{(p\varpi)^{\frac{1}{2}}}{c^2} \left[\frac{\delta \Delta}{\delta - \Delta}\right]^{\frac{1}{2}}. \tag{17}$$

12. This formula, already very simple, may be still further simplified by remarking that for the usual values of δ and Δ, the function $\left[\frac{\delta \Delta}{\delta - \Delta}\right]^{\frac{1}{2}}$ is very nearly proportional to Δ. Adopting this interpolation, which is quite sufficient for practice, the approximate formula for pressures is

$$P = Ka^2 \Delta \frac{(p\varpi)^{\frac{1}{2}}}{c^2}, \tag{18}$$

and under this form we shall employ it in the researches to follow.

13. *Numerical determination of K.*—Supposing the characteristic a of a powder to be known, it is sufficient to measure the maximum pressure for that powder given under a single condition of firing, in order to find from formula (18) the numerical value of the constant K.

For example, this determination can be made for the standard powder referred to in No. 9. This powder fired under the conditions defined in the table of the article before cited, has given pressures which, expressed in kilograms per square centimeter, are equal to

2290 kil. in the 24 cent. gun,
1140 " " 14 " .

(These numbers are the means of the readings of three crusher gauges placed in the powder chamber.)

Consequently with our units the values of P in these two cases are respectively equal to 229000 and 114000. With these numbers two

values of K are deduced which differ but little from each other, and which give a remarkable verification of formula (18).

14. These values may be calculated in two ways:

1st. By supposing, as in Art. 9, $\alpha = 1$ for the standard powder. We find K to be:

$$(19) \qquad \left\{ \begin{array}{l} \text{For the 24 cent. gun} \quad . \quad K = 26760 \\ \text{“ 14 “ “} \quad . \quad K = 27360 \end{array} \right\} \text{mean: } 27060.$$

2d. By supposing, as in Art. 10, $\alpha = 1331$ for the standard powder. Under this hypothesis, we are led to divide by the square of a_0 the preceding value of K, which gives for the new value of K

$$(20) \qquad K = 0.0153.$$

The corresponding formulæ for P are:

$$(21) \qquad P = 27060 a^2 \Delta \frac{(p\varpi)^{\frac{1}{2}}}{c^2},$$

$$(22) \qquad P = 0.0153 a^2 \Delta \frac{(p\varpi)^{\frac{1}{2}}}{c^2}.$$

15. For reasons given in Art. 10, we are led to prefer and to adopt formula (22).

In the preceding formulæ for pressures as in those for velocities the units chosen are the decimeter, the kilogram, and the second. It is necessary then to divide the results of formula (22) by 100, if we wish, as is customary, to express the pressure in kilograms per square centimeter.

CHAPTER II.

Applications and Numerical Verifications.

16. In this chapter we shall show how by the preceding formulæ we may determine the characteristics of various powders, and make use of these determinations to calculate *à priori* the velocities and pressures in any arm whatever. The results will be compared with the results of direct measurement.

17. *Experimental determination of the characteristics.*—For the same powder, characterized by a system of values α, β, the initial velocity depends upon the five quantities,

$$c, u, p, \varpi, \Delta,$$

and is connected with them by (15).

It is evident from the form of this equation, that α and β will become known from two velocities determined by experiment with different systems of particular values of the variables, provided that one at least of the first three shall have different values in the two experiments.

Put for shortness,

$$(23)\qquad 1.569\left(\frac{\varpi}{p}\right)^{\frac{1}{4}}\left(\frac{\varpi}{c^2}\right)^{\frac{1}{10}}\Delta^{\frac{1}{4}}u^{\frac{4}{10}}=\frac{1}{X},\quad 0.00795\,\frac{(pu)^{\frac{1}{2}}}{c}=Y;$$

equation (15) may then be written

$$Xv=\alpha-\alpha\beta Y.$$

Calling X_1, Y_1; X_2, Y_2, two systems of values of X and Y, and v_1 and v_2 the corresponding measured velocities, we have

$$(24)\qquad \begin{cases} X_1v_1=\alpha-\alpha\beta Y_1, \\ X_2v_2=\alpha-\alpha\beta Y_2; \end{cases}$$

from which $\alpha\beta$ and α may be found by the relations,

$$\alpha\beta=\frac{X_2v_2-X_1v_1}{Y_1-Y_2},$$

$$\alpha=X_1v_1+\alpha\beta Y_1.$$

18. *On the error committed in the determination of the characteristics.*—It is easy to determine the effect of an error in the measurement of the velocities upon the determination of the characteristics. Consider the values of $\alpha\beta$ and α in the form

$$\alpha\beta=\frac{X_2v_2-X_1v_1}{Y_1-Y_2},$$

$$\alpha=\frac{Y_1X_2v_2-Y_2X_1v_1}{Y_1-Y_2}.$$

Let ε be the absolute value of the greatest error which is to be apprehended in the measurement of the velocities. The error corresponding in $\alpha\beta$ and α is a maximum when the error ε affects v_2 and v_1 in opposite senses. We have, consequently, the two limits,

$$(25)\qquad \varepsilon_1=\frac{X_2+X_1}{Y_1+Y_2}\varepsilon,\quad \varepsilon_2=\frac{Y_1X_2+Y_2X_1}{Y_1-Y_2}\varepsilon.$$

The first concerns $\alpha\beta$, and the second α. It is evident from these values that Y_1-Y_2 should be made as large as possible by the choice of proper conditions of fire.

19. *Determination of the characteristics by firing guns of* 14 *and* 24 *centimeters.*—Firing these guns, in the conditions specified in No. 9, will answer very well for this determination. The numerical values of X and Y, calculated by (23), making use of the elements in the table of No. 9, are as follows:

$$\text{24 cent. gun} \begin{cases} X_1 = 0.20287, \\ Y_1 = 0.24362, \end{cases}$$

$$\text{14 cent. gun} \begin{cases} X_2 = 0.24735, \\ Y_2 = 0.13534. \end{cases}$$

The limits of error are given by the formulæ,

$$\varepsilon_1 = 3.99\varepsilon, \qquad \varepsilon_2 = 0.810\varepsilon;$$

in which ε, as well as the velocities, should be expressed in decimeters.

The following table shows the results of determinations made by this method upon various lots of powder tried by the committee at Gavre. These powders were all mixed in the proportions used in England (75 saltpetre, 10 sulphur, and 15 charcoal); and were fabricated with *meules et presses.*

Name of Powder.	Velocity with gun of 24 cent.	Velocity with gun of 14 cent.	Values of α	Values of β
A_3B.	439.7	441.6	1340	1.376
A_3S_1, No. 1, . .	453.8	464.1	1431	1.464
" " 2, . .	451.0	457.4	1401	1.425
" " 3, . .	446.7	447.8	1359	1.358
" " 4, . .	431.1	432.3	1312	1.370
" " 5, . .	412.2	478.2	1059	0.863

20. *Determination of the characteristics by firing with different projectiles.*—In the absence of guns of different calibres the characteristics may be determined by firing different projectiles in the same piece. We might, for example, measure the velocities of the 96 and 144 kilo. projectiles from the 24 cent. gun, the other elements being as defined in No. 9. We have in this case,

$$\text{96 kilo. projectile,} \begin{cases} X_1 = 0.18332, \\ Y_1 = 0.19890. \end{cases}$$

$$\text{144 kilo. projectile,} \begin{cases} X_2 = 0.20287, \\ Y_2 = 0.24362; \end{cases}$$

the limits of error are

$$\varepsilon_1 = 8.64\varepsilon, \qquad \varepsilon_2 = 1.90\varepsilon;$$

these are less favorable to accuracy than in the first method.

21. *Approximate determination of the force and velocity of combustion of powder.*—When the form of the grain is well defined, a and λ may be determined from (9). We may then determine f and τ, after the determination of α and β, by the formulæ

$$\tau = \frac{\lambda}{\beta}, \quad f = \frac{\tau a^2}{a}. \tag{26}$$

The time of combustion τ is the time in free air. We have for the corresponding velocity of combustion,

$$w = \frac{e}{2\tau}, \tag{27}$$

e being the least dimension of the grain.

Let us apply this to the powders treated of in No. 19. Their grain is very nearly a parallelopipedon with square base. Their least dimension (thickness) is exactly regulated by the caking, and can be directly measured. The ratio of the other dimensions to the last can be calculated as a function:

1st. Of the thickness e;

2d. Of the number of grains in a kilogram N;

3d. Of the density δ,

by the formula (Part III, No. 45),

$$x = (\delta N)^{\frac{1}{3}} e^{\frac{2}{3}}, \tag{28}$$

and the coefficients a and λ by the relations (Part III, No. 41),

$$a = 1 + 2x, \quad \lambda = \frac{2x + x^2}{1 + 2x}. \tag{29}$$

The quantities e, N, and δ for the powders considered are known, and the characteristics having been determined, we may calculate f and w. The following table gives the elements and results of the calculation:

Name of Powder.	e dec.	N	δ	x	a	λ	τ sec.	f	w dec.
A_8B	.128	105	1.810	.631	2.262	.734	.534	424000	.120
A_8S_1, No. 1	.130	116	1.776	.673	2.346	.767	.524	457000	.124
" No. 2	.130	114	1.794	.670	2.340	.765	.537	451000	.121
" No. 3	.130	113	1.805	.670	2.340	.765	.560	442000	.116
" No. 4	.130	107	1.807	.652	2.304	.750	.545	410000	.119
" No. 5	.130	104	1.813	.644	2.228	.739	.846	424000	.076

22. *Calculation of the characteristics of a powder from its physical properties.*—The force of a powder and the velocity of its combustion for a given density depend exclusively upon the process of its

manufacture. If, then, these have been determined for a particular powder, they may be used with any other powder of the same fabrication, however it may differ as to form and dimensions of grain.

When the density of the powder changes, we may calculate the velocity of its combustion approximately by assuming, as indicated by Piobert's experiments, that the velocity varies inversely as the density. We may thus, for all powders of a given fabrication, determine *à priori* the approximate values of f and w. From an examination of the form and dimensions of the grain, the values of τ, α and λ, may then be determined by (27) and (29) of the preceding paragraph.

We have thus all the elements upon which the characteristics depend (9).

22½. *Characteristics of some common powders.*—The following table gives the results of the application of the methods stated to some common powders:

Name of Powder.	N	δ	w dec.	f	ϵ dec.	x	a	λ	τ	α	β
W(20-25)	110	1.800	.0972	431000	.172	1.000	3.000	1.000	.885	1209	1.130
W(16-20)	230	1.775	.0986	431000	.167	1.000	3.000	1.000	.848	1234	1.179
W(13-16)	350	1.750	.1000	431000	.146	1.000	3.000	1.000	.730	1331	1.370
W(10-13)	600	1.715	.1020	431000	.123	1.000	3.000	1.000	.602	1465	1.660
SP_2 . . .	105	1.810	.1200	424000	.128	.631	2.262	.734	.534	1340	1.376
SP_1 . . .	350	1.790	.1213	424000	.100	.792	2.584	.856	.412	1630	2.076
C_3. . . .	625	1.760	.1227	424000	.080	.750	2.500	.825	.324	1809	2.545
C_1. . . .	1750	1.745	.1245	424000	.065	.916	2.832	.943	.261	2147	3.612

Remarks.—The powders marked W were from the Wetteren factory. The value $f=431000$ is adopted for them; they are described in No. 9.

The velocities of combustion for them are calculated by the formula

$$w=\frac{.1750}{\delta},$$

the constant being so taken as to give for $\delta=1.750$ the value $w=.100$. The grains of the powder W(20-25) are considered as cubes; those of the three others being very irregular, were taken as spheres.

The fabrication of the four other powders in the table is the same as that of the powder A_2B in No. 21. Consequently, the value $f=424000$ was taken for them; and their velocities of combustion were calculated by the formula

$$w=\frac{.2172}{\delta},$$

the constant being so taken as to give for the value $\delta = 1.810$, $w = .120$, corresponding to A_3B. The grains of these powders are considered parallelopipedons with square bases.

23. *Calculation of initial velocities and comparison with the measured results.*—The following table gives the results of this calculation for guns of various calibres, and the corresponding measured velocities:

Name of Gun.	Kind of Powder.	c	u	f Kil.	ϖ	Δ	Velocities. Measured. Meters.	Velocities. Calculated. Meters.	Differences.
32 cent.	W (20-35)	3.220	42.19	352.0	65.	0.786	415.6	414.8	+ 0.8
"	"	"	"	286.5	62.	0.749	451.0	445.7	+ 5.3
"	"	"	"	286.5	61.	0.730	442.5	441.3	+ 1.2
27 cent.	W (20-25)	2.764	40.97	217.0	42.	0.800	431.9	432.0	— 0.1
"	"	"	"	180.0	42.	0.800	470.0	470.3	— 0.3
24 cent.	W (13-16)	2.420	34.45	144.0	28.	0.798	432.0	431.1	+ 0.9
"	"	"	"	Ogival.	24.	0.684	393.2	393.1	+ 0.1
"	"	"	"	"	20.	0.570	352.0	352.4	— 0.4
"	"	"	"	"	16.	0.456	306.6	308.3	— 1.7
"	"	"	"	144.	28.	0.798	428.8	431.1	— 2.3
"	"	"	"	cylindri'l.	24.	0.684	390.5	393.1	— 2.6
"	"	"	"	"	20.	0.570	352.0	352.4	— 0.4
"	"	"	"	"	16.	0.456	303.4	308.3	— 4.9
"	"	"	"	120.	28.	0.798	467.4	469.5	— 2.1
"	"	"	"	"	24.	0.684	427.2	428.1	— 0.9
"	"	"	"	"	20.	0.570	379.5	383.7	— 4.2
"	"	"	"	"	16.	0.456	332.5	335.7	— 3.4
"	"	"	"	96.	28.	0.790	512.5	517.8	— 5.3
"	"	"	"	"	24.	0.684	469.2	472.2	— 3.0
"	"	"	"	"	20.	0.570	420.6	423.2	— 2.7
"	"	"	"	"	16.	0.456	363.0	370.2	— 7.2
"	"	"	38.19	144.	28.	0.789	437.1	437.1	— 7.2
"	"	"	40.36	144.	28.	0.778	434.9	439.0	— 4.1
"	"	"	"	"	24.	0.667	396.6	400.1	— 3.5
"	"	"	"	"	20.	0.556	360.4	358.8	+ 1.6
"	"	"	"	"	16.	0.444	312.3	313.6	— 1.3
19 cent.	"	1.94	30.02	62.5	15.0	0.991	492.2	494.1	— 1.9
"	"	"	"	"	12.5	0.826	444.6	443.0	+ 1.6
"	"	"	"	"	10.0	0.661	393.3	387.5	+ 5.8
"	"	"	"	76.0	15.0	0.991	453.5	454.9	— 1.4
"	"	"	"	"	14.5	0.958	446.4	446.0	+ 0.4
"	"	"	"	"	13.5	0.892	428.7	427.2	+ 1.5
"	"	"	"	"	12.5	0.826	411.8	407.9	+ 3.9
15.5 cent.	SP_1	1.564	24.94	31.8	4.9	0.751	400.2	406.8	— 6.6
"	"	"	"	36.3	4.26	0.653	349.8	351.4	— 1.6
14 cent.	W (13-16)	1.40	26.97	18.65	4.0	0.958	472.0	466.7	+ 5.3
"	"	"	"	"	3.5	0.838	431.2	429.8	+ 1.4
"	"	"	"	"	3.0	0.719	392.0	391.9	+ 0.1
"	"	"	"	23.30	4.0	0.958	428.0	429.3	— 1.3
"	"	"	"	"	3.5	0.838	400.5	396.4	+ 4.1
"	"	"	"	"	3.0	0.719	366.1	361.5	+ 4.6
"	"	"	27.04	21.0	4.0	0.891	438.4	438.4	+ 4.6
13.8 cent.	SP_1	1.390	24.55	23.57	3.8	0.647	400.1	399.6	+ 0.5
"	"	"	"	"	3.46	0.589	377.4	377.8	— 0.4
"	"	"	"	"	2.8	0.477	333.4	332.7	+ 0.7

Name of Gun.	Kind of Powder.	c	u	$\frac{p}{\text{Kil.}}$	ϖ	Δ	Velocities. Measured. Meters.	Velocities. Calculated. Meters.	Differences.
12 cent.	SP_1	1.210	19.02	16.32	2.8	0.754	403.3	403.9	— 0.6
"	"	"	"	"	1.74	0.468	299.3	303.6	— 4.3
"	"	"	"	18.25	2.42	0.651	352.2	353.2	— 1.0
"	"	"	"	"	1.85	0.498	296.3	300.6	— 4.3
9.5 cent.	C_2	0.960	23.96	9.13	1.73	0.609	432.1	432.0	+ 0.1
"	"	"	"	8.445	1.63	0.574	426.0	428.5	— 2.5
9 cent.	C_2	0.907	17.98	8.0	1.50	0.888	448.6	447.7	+ 0.9
"	"	"	"	6.5	1.50	0.888	484.7	488.7	— 4.0
"	"	0.911	24.0	8.76	1.60	0.629	426.7	426.4	+ 0.3
"	"	"	"	8.19	1.525	0.597	424.7	427.1	— 2.4
"	"	"	"	7.78	1.47	0.580	422.5	420.3	+ 2.2
"	"	0.909	21.63	10.9	2.40	0.734	448.7	450.6	— 1.9
"	"	"	"	10.9	2.00	0.612	399.0	403.9	— 4.9
"	"	"	"	9.6	2.50	0.765	484.7	492.0	— 7.3
"	"	"	"	9.6	2.22	0.679	449.0	458.1	— 9.1
"	"	"	"	9.6	1.84	0.563	400.4	409.3	— 8.9
"	C_1	0.907	17.98	8.0	1.52	0.899	448.0	453.5	— 5.5
"	"	"	"	8.0	1.40	0.828	434.0	431.7	+ 2.3
"	"	0.9095	18.76	8.0	2.10	0.758	500.6	489.1	+ 11.5
"	"	"	"	8.0	1.94	0.700	475.7	466.5	+ 8.7
8 cent.	"	0.811	16.90	6.0	1.54	0.866	501.0	502.0	— 1.0
"	"	"	"	5.6	1.47	0.826	500.0	505.8	— 5.8
"	"	0.805	19.28	5.5	1.60	0.899	528.6	524.9	+ 3.7
"	"	"	"	5.5	1.40	0.828	484.7	484.5	+ 0.2
7.5 cent.	"	0.750	18.46	5.0	1.250	0.709	484.0	480.0	+ 4.
"	"	"	"	5.0	1.425	0.808	528.0	520.0	+ 8.
"	"	"	"	5.0	1.500	0.850	551.0	546.0	+ 5.
"	"	0.750	15.46	5.0	1.250	0.709	470.0	469.0	+ 1.
"	"	"	"	5.0	1.425	0.808	510.0	508.0	+ 2.
"	"	"	"	5.0	1.500	0.850	529.0	524.0	+ 5.

The preceding table shows that the theoretical formula gives the velocity with very satisfactory exactness in 75 very different conditions of fire in guns whose calibre varies from 320 to 75 millimeters. The mean error for the velocities varying from 300 to 550 meters is less than 3 meters.

It is important to notice that the varying ballistic qualities of the powders of different fabrication taken have doubtless caused some of the discordances in the table.

24. *Fire with the* 16 *cent. gun.*—The following, for example, is a case where the difference between the calculated and measured velocity should, it would appear, be attributed to variations in the qualities of the powder. It is the case of the use of Wetteren powder (13 to 16 mill.) in the 16 cent. gun, model 1870, short.

The principal dimensions of the gun are the following:

Calibre, ... $c = 1.661$

Motion of shot, ... $u = 27.28$

In the following table the calculated and measured velocities will be found. The powder is supposed to have the characteristics of the standard powder.

Name of Gun.	c.	u.	β.	ϖ	Δ	Velocities. Measured.	Velocities. Calculated.	Differences.
Gun of 16 cent., model 1870.	1.661	27.28	31	5.	0.540	379.4	365.6	+ 13.8
	"	"	"	7.	0.756	467.7	447.4	+ 20.3
	"	"	"	9.	0.972	537.5	520.1	+ 17.4
	"	"	"	9.5	1.026	554.1	537.4	+ 16.7
	"	"	38.25	5.	0.540	351.8	337.9	+ 13.9
	"	"	"	7.	0.756	426.0	413.5	+ 12.5
	"	"	"	9.	0.972	495.0	480.8	+ 14.2
	"	"	"	9.5	1.026	508.3	496.7	+ 11.6
	"	"	45	5.	0.540	326.4	317.1	+ 9.3
	"	"	"	7.	0.756	397.8	388.1	+ 9.7
	"	"	"	9.	0.972	461.8	451.1	+ 10.7
	"	"	"	9.5	1.026	475.2	466.1	+ 9.1

These differences are large, and would throw doubt upon the formula did not the *procès-verbaux* of the experiments show that the powder was different from the standard powder.

In fact, this powder (taken from the lot number 15) gave to the 24 and 14 cent. guns in the conditions specified in No. 9, velocities respectively of 435 and 456 meters, instead of 437 and 438, which were those given by the standard powder.

The characteristics of lot No. 15 deduced from the velocities by the method of No. 19 are

$$\alpha = 1429, \quad \beta = 1.570;$$

these differ considerably from those of the standard powder, which are

$$\alpha = 1331, \quad \beta = 1.370.$$

These new values have been used in the calculation of the velocities in the following table, whose accuracy is very satisfactory.

Name of Gun.	c	u	β	ϖ	Δ	Velocities. Measured.	Velocities. Calculated.	Differences.
Gun of 16 cent., model 1870.	1.661	27.28	31.	5.	0.540	379.4	379.0	+ 0.4
	"	"	"	7.	0.756	467.7	463.8	+ 3.9
	"	"	"	9.	0.972	537.5	539.1	− 1.6
	"	"	"	9.5	1.026	554.1	557.1	− 3.0
	"	"	38.25	5.	0.540	351.8	348.6	+ 3.2
	"	"	"	7.	0.756	426.0	426.6	− 0.6
	"	"	"	9.	0.972	495.0	495.9	− 0.9
	"	"	"	9.5	1.026	508.3	512.4	− 4.1
	"	"	45.	5.	0.540	326.4	325.6	+ 0.8
	"	"	"	7.	0.756	397.8	398.4	− 0.6
	"	"	"	9.	0.972	461.8	463.2	− 1.4
	"	"	"	9.5	1.026	475.2	478.5	− 3.3

25. It will be interesting to ascertain the nature of the change undergone by this powder; the considerations set forth in No. 21 will enable us to do this. If, by (26), we derive from the values of the characteristics those of the force and time of combustion, we find for the powder considered,

$$f = 433000, \quad \tau = 0.637.$$

For the standard powder we have

$$f = 431000, \quad \tau = 0.730.$$

The values of f do not differ sensibly, but the values of τ do. The velocity of combustion has therefore been altered by some circumstance of fabrication.

26. *Calculation of pressures.*—The following are applications of (22) to the calculation of the maximum pressure; this requires that α shall be known.

This element having been determined by the five lots of A_2S powder considered in No. 19, we may calculate the pressure produced by this powder in the 24 centimeter gun in the ordinary conditions; and, as these pressures have generally been measured by *manomètres à écrasement*, the calculated results may be compared with these. The following table contains the data and results:

Name of Powder.	c	p	ϖ	Δ	Measured Velocities.	Pressures.	
						Measured.	Calculated.
A_2S' No. 1	2.42	144	28	0.789	453.8	2790	2680
" No. 2	"	"	"	"	451.0	2480	2570
" No. 3	"	"	"	"	446.7	2330	2420
" No. 4	"	"	"	"	431.1	2010	2250
" No. 5	"	"	"	"	412.2	1440	1470

The agreement of the calculated and measured pressures is not so close as in the case of the velocities; but it must be remembered that the instruments with which the pressures are measured are not accurate. In fact, it is evident, from the preceding table, that the formula represents, generally, the variations of pressure with an accuracy which, if it is always maintained, is sufficient for practical purposes. This last point can only be satisfactorily determined when the measurements of internal pressures shall be more varied and more numerous than at present.

27. We resume in the following table the results of calculation in some of the conditions of fire which form the object of No. 19.

The conditions are those which, for each gun, correspond to the greatest simultaneous values of the weights of projectile and charge.

Name of Gun.	Name of Powder.	c	u	p	ϖ	Δ	Observed velocity.	Calculated pressure.
32 cent.	W(20-25)	3.220	42.19	357.	65.	0.786	415.6	2560
27 "	"	2.764	40.97	217.	42.	0.800	431.9	2240
24 "	W(13-16)	2.420	38.19	144.	28.	0.789	437.1	2320
19 "	"	1.940	30.02	76.	15.	0.991	454.9	2410
15.5 "	SP_1 . .	1.564	24.94	36.3	4.26	0.653	349.8	1350
14 "	W(13-16)	1.400	26.97	23.3	4.	0.958	428.0	1280
13.8 "	SP_1 . .	1.390	24.55	23.57	3.80	0.647	400.1	1300
12 "	" . . .	1.210	19.025	18.25	2.42	0.651	352.2	1200
9.5 "	C_3 . . .	0.960	23.96	9.13	1.73	0.609	432.1	1320
9 "	" . . .	0.907	17.98	8.	1.50	0.888	448.6	1870
9 "	" . . .	0.911	24.00	8.76	1.60	0.629	426.7	1420
9 "	" . . .	0.909	21.63	10.90	2.40	0.734	448.7	2280
9 "	C_1 . . .	0.907	17.98	8.	1.52	0.899	448.0	2680
9 "	" . . .	0.9095	18.76	8.	2.10	0.758	500.6	2650
8 "	" . . .	0.811	16.90	6.	1.54	0.866	501.0	2820
8 "	" . . .	0.805	19.28	5.50	1.60	0.899	528.6	2900
7.5 "	" . . .	0.750	18.46	5.	1.50	0.850	551.0	2920

CHAPTER III.

I.—Maximum of Velocities.

28. In Art. 6 the initial velocity is given as a function of the elements of firing by the formula:

$$v = A\left(\frac{fa}{\tau}\right)^{\frac{1}{2}}\left(\frac{\varpi}{p}\right)^{\frac{1}{4}}\left(\frac{\varpi}{c^3}\right)^{\frac{1}{10}}\Delta^{\frac{1}{4}}u^{\frac{4}{10}}\left[1 - B\frac{\lambda}{\tau}\frac{(pu)^{\frac{1}{2}}}{c}\right],$$

A and B being two constants. The signification of the other letters is that given in Articles 2, 5, and 7 of the first Chapter. Now if in this formula we make τ a variable, all the other elements being constant, the velocity will pass through a maximum.

29. Put for brevity:

$$\varphi(\tau) = \tau^{-\frac{1}{2}} - B\frac{\lambda(pu)^{\frac{1}{2}}}{c}\tau^{-\frac{3}{2}} \tag{30}$$

and we have:

$$v = A(fa)^{\frac{1}{2}}\left(\frac{\varpi}{p}\right)^{\frac{1}{4}}\left(\frac{\varpi}{c^3}\right)^{\frac{1}{10}}\Delta^{\frac{1}{4}}u^{\frac{4}{10}}\varphi(\tau). \tag{31}$$

The value of τ corresponding to the maximum of φ, and therefore of v, is given by the condition $\varphi'(\tau)=0$, that is to say, by the relation,

$$B\frac{\lambda(pu)^{\frac{1}{2}}}{\tau c}=\frac{1}{3}, \tag{32}$$

denoting that value by τ_1 we have

$$\tau_1=3B\frac{\lambda(pu)^{\frac{1}{2}}}{c}. \tag{33}$$

The corresponding velocity is then deduced from the relation

$$v_1=\frac{2}{3}A\left(\frac{fa}{\tau_1}\right)^{\frac{1}{2}}\left(\frac{\varpi}{p}\right)^{\frac{1}{2}}\left(\frac{\varpi}{c^2}\right)^{\frac{1}{10}}\Delta^{\frac{1}{4}}u^{\frac{4}{10}}, \tag{34}$$

or better still, replacing τ_1 by its value (33),

$$v_1=A_1\left(\frac{fa}{\lambda}\right)^{\frac{1}{2}}\frac{\varpi^{\frac{7}{10}}c^{\frac{3}{10}}\Delta^{\frac{1}{4}}u^{\frac{3}{10}}}{p^{\frac{1}{2}}}, \tag{35}$$

the value of A_1 being (1)

$$A_1=\frac{2}{3}A(3B)^{-\frac{1}{2}}.$$

30. The consideration of the theoretical maximum permits us to express the formula for velocities in another form. For a powder nearly like the maximum let

$$\tau=\tau_1+\theta.$$

We have for the corresponding value of the function $\varphi(\tau)$:

$$\varphi(\tau)=\varphi(\tau_1+\theta)=\varphi(\tau_1)\theta+\varphi'(\tau_1)\theta+\varphi''(\tau_1)\frac{\theta^2}{2}+\dots$$

or simply:

$$\varphi(\tau)=\varphi(\tau_1)+\varphi''(\tau_1)\frac{\theta^2}{2},$$

taking into account the condition $\varphi'(\tau_1)=0$ and neglecting terms of an order greater than the second.

This being granted, we find from formula (30):

$$\varphi''(\tau)=\frac{3}{4}\tau^{-\frac{5}{2}}\left[1-5B\frac{\lambda(pu)^{\frac{1}{2}}}{\tau c}\right].$$

Replacing τ by τ_1 and taking into account the condition (32) which is then satisfied, it becomes:

$$\varphi''(\tau_1)=-\frac{1}{2}\tau_1^{-\frac{5}{2}}.$$

Under the same hypothesis, formula (30) gives

$$\varphi(\tau_1)=\frac{2}{3}\tau_1^{-\frac{1}{2}}.$$

We can then write: $\varphi''(\tau_1) = -\frac{3}{4}\cdot\frac{\varphi(\tau_1)}{\tau_1^2}$, and consequently

$$\varphi(\tau) = \varphi(\tau_1)\left[1 - \frac{3}{4}\frac{\theta^2}{\tau_1^2}\right],$$

whence the formula for velocities:

$$v = v_1\left[1 - \frac{3}{4}\frac{\theta^2}{\tau_1^2}\right]. \tag{36}$$

v_1 and τ_1 being the values of (35) and (33). Formula (36) shows that the velocity is very little modified when τ varies in the neighborhood of the value of the maximum. If, for example, we suppose $\frac{\theta}{\tau_1} = 0.2$, that is to say, if the difference between τ and τ_1 is $\frac{1}{5}$ of the value of τ_1, the corresponding difference between the velocities v and v_1 is only $\frac{3}{100}$ of the value of v_1.

32. *Ballistic coefficient.*—It results from formula (35) that for given values of the various elements of firing, the greatest velocity that a powder can produce is proportional to $\left(\frac{fa}{\lambda}\right)^{\frac{1}{2}}$. This factor, which has been called the *ballistic coefficient*, depends at the same time on the force of the powder and the form of the grain.

If the force is constant the ballistic coefficient is proportional to $\left(\frac{a}{\lambda}\right)^{\frac{1}{2}}$. It depends only on the form of the grain.

When the grain is spherical or cubical we have $a = 3$ and $\lambda = 1$. If then we designate by C the ballistic coefficient of a powder referred to that of a spherical or cubical powder of the same force, we can put:

$$C = \left(\frac{a}{3\lambda}\right)^{\frac{1}{2}}.$$

The value of C is greater than 1 for grains of the form of a parallelopipedon or a pierced cylinder. We can, then, by the employment of these grains, increase the velocity without changing the other elements of firing. But in practice this increase cannot pass a certain limit on account of the corresponding increase of the maximum pressure, which varies as the square of the ballistic coefficient, and rapidly approaches the limit imposed by the resistance of the piece.

33. Several experimental facts, notably those of Maguin, demonstrate that there exists for each gun and in each condition of firing, a size of grain and consequently a duration of combustion of the grain, which gives a maximum of velocity.

It remains to verify if this maximum is shown in the conditions indicated by theory. The following is a case in which theory and experiment seem to be in accord.

If we consider the firing of a 24 cent. gun under the habitual conditions of proof, and if we calculate the value of τ which for a spherical or cubical grain correspond to the maximum, making in formula (33) $\lambda = 1$, $c = 2.42$, $u = 38.19$, $p = 144$, we find

$$\tau_1 = 0.731,$$

a value exactly equal to that given in the table of No. 22 for the standard powder. Wetteren powder (grains of 13 to 16 mill.) is then the maximum for the 24 cent. gun. This result seems confirmed by experiment, since the sample No. 15 duration of combustion 0.637, and less than that of the standard powder, gave only 435 meters initial velocity as compared to 437 meters given by the standard powder.

34. It would be an advantage to adopt a maximum powder, not only because the charge would thus be better utilized, but also for the reason that the chance variations which are made in the manufacture of the powder, affecting the velocity of combustion, would not have a sensible influence on the initial velocity, which assures regularity of fire.

This remark explains the relatively small differences shown in the proof of samples of powder for the 24 cent. gun. The differences should be more marked in the case of the 14 cent. gun, for which the maximum powder is very different from the standard powder of 13 to 16 millimeters. (In the case of the 14 cent. gun, we have $\lambda = 1$, $C = 1.40$, $u = 27.04$, $p = 21$; the corresponding value of τ_1, calculated by formula (33), is

$$\tau_1 = 0.406.$$

It should be remarked, however, that the employment of a maximum powder gives generally very high interior pressures, and this inconvenience is also more serious since irregularities in the fabrication can in this case produce great variations of pressure without sensibly modifying the velocity. This particularity is found from the special form of the functions which represent the velocity and the maximum pressure. The velocity is the difference of two terms increasing together as τ diminishes, so that their variations compensate near the maximum. The pressure is represented, however, by a monomial

formula, and it varies rapidly in the inverse ratio of the period of combustion.

Example: The sample of Wetteren powder No. 15 already cited gives, in the 24 cent. gun, 435 meters initial velocity. The initial velocity given by the standard powder is 437 meters. The calculated corresponding pressures are 2660 kil. for sample No. 15, and 2330 kil. for the standard powder.

Sample No. 15 is then notably more violent than the standard powder, and also the initial velocity given is a little less.

II.—Maximum of Living Force.

35. When we increase the mass m of the projectile, the other elements remaining constant, the velocity v diminishes in such a manner that the living force mv^2 increases in general. Experiment indicates, however, that this augmentation is not indefinite, and that the living force attains a maximum.

This remarkable fact, which M. Navez was the first to assert, does not accord with ballistic formulæ generally used. It is, however, very well explained by the theoretical formula. This formula indicates, in fact, that the product of the velocity by the square root of the weight of the projectile passes through a maximum.

Put for brevity

$$\psi(p) = p^{\frac{1}{2}} - B\frac{\lambda u^{\frac{1}{2}}}{\tau c}p^{\frac{1}{2}}. \tag{37}$$

We have from the formula for velocities

$$p^{\frac{1}{2}}v = A\left(\frac{fa}{\tau}\right)^{\frac{1}{2}}\varpi^{\frac{1}{2}}\left(\frac{\varpi}{c^2}\right)^{\frac{1}{1-\sigma}}\Delta^{\frac{1}{2}}u^{\frac{\lambda}{1-\sigma}}\psi(p). \tag{38}$$

The maximum of this expression, since p varies, corresponds to the maximum of ψ, and is given by the condition $\psi'(p) = 0$, that is to say, in making the calculation by the relation

$$B\frac{\lambda(pu)^{\frac{1}{2}}}{\tau c} = \frac{1}{3}.$$

This relation is the same as (32), which we have found to be the relation expressing the duration of the combustion of a grain which gives a maximum velocity.

Consequently, in establishing this relation between the quantities λ, τ, p, u and c, we make a maximum :

1st. The velocity of the projectile with respect to the variations of the duration of combustion of the powder.

2d. The living force of the projectile with respect to the variations of its weight. This relation is satisfied, according to No. 33, in the firing of Wetteren standard powder (grains of 13 to 16 mill.) in the 24 centimeter gun with a projectile of 144 kilograms.

The result is, on account of the relations adopted, the projectile weighing 144 kilograms corresponds to the maximum for useful effect.

III.—Influence of the Various Elements of Firing on Velocity and Pressure.

36. In a cannon we may consider the quantities

$$c, u, p,$$

(calibre, length of path, and weight of projectile) as constant or given quantities. The variables with which in connection with the constant quantities we wish to obtain a given velocity, are

$$f, a, \lambda, \tau, \varpi, \Delta,$$

and there may be an infinite number of combinations made with these variables giving the same velocity with different maximum pressures or the same pressure with different velocities.

It has been shown (Part III, Art. 57) how we can, with fixed values of ϖ and Δ, lower the pressure and retain the velocity, or increase the velocity and retain the same pressure by a proper choice of the variables,

$$f, a, \lambda, \tau,$$

which refer exclusively to the powder.

37. We may consider as known also f, a, λ, that is to say, the force of the powder and the form of the grain, and evaluate the influence of ϖ, Δ, and τ on the values of the velocity and maximum pressure. With this in view, consider formula (31) for velocities. The factor $\varphi(\tau)$ is given by equation (30), which we may write, taking into account equation (33), in an abbreviated form, thus:

$$\varphi(\tau) = \frac{3\tau - \tau_1}{3\tau^{\frac{3}{2}}}, \tag{39}$$

τ_1 being the value of τ corresponding to the maximum velocity with the form of grain adopted.

If, then, we suppose the quantities ϖ, Δ and τ to be the only variables, we may put

$$v = K\varpi^{\frac{1}{6}}\Delta^{\frac{1}{2}}\tau^{-\frac{1}{2}}(3\tau - \tau_1),$$

K' denoting a constant. The corresponding value of the maximum pressure is, according to equation (18):

$$P = K' \frac{\varpi^{\frac{1}{2}} \Delta}{\tau}.$$

38. We wish to find the variation of v and P for very small increments $d\varpi$, $d\Delta$, $d\tau$, of the variables. We find very easily:

$$\frac{dv}{v} = \frac{7}{20}\frac{d\varpi}{\varpi} + \frac{1}{4}\frac{d\Delta}{\Delta} - \frac{3}{2}\frac{\tau - \tau_1}{3\tau - \tau_1}\cdot\frac{d\tau}{\tau}, \tag{40}$$

$$\frac{dP}{P} = \frac{1}{2}\cdot\frac{d\varpi}{\varpi} + \frac{d\Delta}{\Delta} - \frac{d\tau}{\tau}. \tag{41}$$

The influence of each variable on the value of the velocity is measured by the coefficient which multiplies the relative variation of each variable in expression (40).

The coefficients of $\frac{d\varpi}{\varpi}$, $\frac{d\Delta}{\Delta}$, and $\frac{d\tau}{\tau}$ are respectively:

$$\frac{7}{20}, \quad \frac{1}{4}, \quad \frac{3}{2}\cdot\frac{\tau - \tau_1}{3\tau - \tau_1}.$$

The third coefficient varies with τ and is equal to zero for $\tau = \tau_1$.

Its value increases with τ, but does not exceed $\frac{1}{4}$ except where τ is greater than $\frac{5}{3}\tau_1$, which is not likely to happen in the habitual conditions of practice.

39. It follows from the preceding that the three coefficients are arranged in the order of their relative value, and that the variations of each of the variables

$$\varpi, \ \Delta, \ \tau$$

have more influence than the one next on its right as arranged above.

On the contrary, the influence of ϖ on the maximum pressure is less than that of Δ or τ.

40. The following tables show, in a particular case, the theoretical values and pressures corresponding to the various values of the elements of firing.

The results show better than can be done by an analytical formula the proper influences of these elements on the effects obtained. These calculations relate to a Krupp gun, calibre 30 centimeters.

The data for this gun may be stated as follows:

Calibre, ...	$C = 3.05$ decimeters
Length of path of projectile .	$u = 44.60$ "
Weight of projectile	$p = 300$ kilograms.

The variables at our disposal are as follows:

f force of the powder,

a, λ characteristics of the form of the grain,

τ duration of the combustion of a grain,

ϖ weight of the charge,

Δ density of loading.

We shall vary them according to the following conditions:

1st. f.—The value of f depends on the composition of the powder and manner of manufacture. We shall adopt the value $f = 424000$, which according to the table of No. 22 is the value for powders of French manufacture.

2d. a, λ.—We will consider three forms of grains—

(A). Cubical or spherical.

(B). Parallelopipedon with square base, the side of the base double the thickness.

(C). Cylinder pierced, the height double the thickness.

The corresponding values of a and λ are the following:

Form of grain.	Values of a	Values of λ
Cubical or Spherical,	3	1
Parallelopipedon,	2	$\frac{1}{2}$
Cylinder, pierced,	$\frac{3}{2}$	$\frac{1}{3}$

3d. τ.—We shall vary τ, by intervals of $\frac{1}{10}$ of a second, starting from particular values which, for the three forms of grain, correspond to the maximum velocity.

These values are calculated by formula (33) with the data c, u, p, shown above.

Form of grain.	Values of τ_1.
Cube or Sphere,	0.904
Parallelopipedon,	0.565
Pierced Cylinder,	0.301

4th. ϖ.—The values given to the weight ϖ of the charge are $\frac{1}{6}$, $\frac{1}{5}$, $\frac{1}{4}$, of the weight of the projectile, or 50, 60, 75 kilograms.

5th. Δ.—Finally, we will consider the values of the density of loading,

1.0 0.9 0.8 0.7

The calculation of velocities and pressures is done without difficulty by the aid of formulæ (15) and (22), in which we recall that the characteristics α, β are respectively

$$\alpha = \left(\frac{fa}{\tau}\right)^{\frac{1}{2}}, \qquad \beta = \frac{\lambda}{\tau}.$$

We have not given in the table velocities greater than 550 meters.

The discussion of the preceding gives rise to some interesting remarks, which may be generalized by remembering the principle of similitude established in a preceding work; but the greater part of the results thus obtained may be deduced with more simplicity and precision from formulæ which will be made the object of a subsequent number. We will therefore abstain from an extensive development of this subject, confining ourselves to the following observations. According to data given by the *Revue d'Artillerie*, the 30.5 cent. Krupp gun has given an initial velocity of 460 meters with a charge of $\frac{1}{3}$ the weight of the projectile, that is, 60 kilograms of prismatic powder, of 25 millimeters.

Now from an examination of the table:

1st. This velocity is impossible, under these conditions, with a powder composed of cubical or spherical grains.

2d. That it may be realized by a powder, grains of the form of a parallelopipedon, with a pressure of nearly 2700 kilograms, a result not assuring, considering the strength of the piece.

3d. That it may be easily obtained with much less pressure, by the use of the pierced cylindrical grain.

We have, for example, a velocity of 460 meters and a pressure of 1800 kilograms very nearly by taking $\Delta = 0.7$ and τ between 0.5 and 0.6. It is easy then to fix the approximate dimensions of the grain. Supposing $\tau = 0.55$, and giving to the powder material a velocity of combustion of 12 millimeters a second, we see that the thickness of the grain would be regulated by the equation $\frac{\varepsilon}{2} = 12 \times 0.55$, whence $\varepsilon = 13.2$ millimeters. The height being double the thickness is then 26.4 millimeters, which seems to accord well with the solution adopted by the German artillery.

CHAPTER IV.

On the greatest Value of the Velocity corresponding to a given Value of the Maximum Pressure.

42. Suppose, for a gun, we make all the elements of firing constant with the exception of the density of loading Δ and the duration of combustion τ, which we consider variable.

TABLE I.—Cubical or Spherical Grain.

($a = 3$, $\lambda = 1$.)

Values of τ.	Charge of $\frac{1}{5}$ ($\varpi = 50$ Kilos.)								Charge of $\frac{1}{5}$ ($\varpi = 60$ Kilos.)								Charge of $\frac{1}{4}$ ($\varpi = 75$ Kilos.)							
	$\Delta = 1$		0.9		0.8		0.7		$\Delta = 1$		0.9		0.8		0.7		$\Delta = 1.0$		0.9		0.8		0.7	
	v	P	v	P	v	P	v	P	v	P	v	P	v	P	v	P	v	P	v	P	v	P	v	P
0.904	428.2	2835	417.0	2252	404.9	2268	391.7	1985	456.4	3105	444.1	2794	431.6	2485	417.4	2174	493.5	3471	480.7	3124	466.8	2777	451.4	2430
1.000	426.6	2563	415.4	2306	403.5	2050	390.2	1780	454.7	2807	442.8	2526	430.0	2245	416.0	1950	491.7	3134	478.9	2823	465.1	2510	449.8	2196
1.10	422.7	2330	411.6	2097	399.8	1864	386.6	1631	450.5	2552	438.7	2296	426.1	2041	412.1	1786	487.2	2852	474.5	2567	460.8	2282	445.7	1997
1.20	417.4	2136	406.5	1922	394.7	1709	381.8	1495	444.8	2339	433.2	2105	420.7	1871	406.9	1637	481.1	2614	468.5	2353	455.0	2092	440.1	1830
1.30	411.2	1971	400.4	1775	388.9	1577	376.1	1380	438.2	2159	426.8	1943	414.5	1728	400.9	1512	473.9	2414	461.5	2172	448.2	1931	433.6	1690
1.40	405.1	1831	394.5	1648	383.1	1464	370.5	1281	431.7	2005	420.4	1804	408.3	1604	394.9	1403	466.9	2241	454.7	2017	441.6	1793	427.1	1569
1.50	397.9	1708	387.5	1537	376.4	1367	364.0	1196	424.1	1871	413.1	1684	401.1	1497	388.0	1310	458.7	1092	446.7	1882	433.8	1673	419.6	1464

TABLE II.—Parallelopipedon with Square Base.

($x = \frac{1}{2}$, $a = 2$, $\lambda = \frac{2}{3}$.)

Values of τ.	Charge of $\frac{1}{5}$ ($\varpi = 50$ Kilos.)								Charge of $\frac{1}{5}$ ($\varpi = 60$ Kilos.)								Charge of $\frac{1}{4}$ ($\varpi = 75$ Kilos.)							
	$\Delta = 1$.		0.9		0.8		0.7		$\Delta = 1$		0.9		0.8		0.7		$\Delta = 1$		0.9		0.8		0.7	
	v	P	v	P	v	P	v	P	v	P	v	P	v	P	v	P	v	P	v	P	v	P	v	P
0.565	442.2	3024	430.6	2722	418.2	2419	404.5	2117	471.3	3565	459.0	2981	445.8	2650	431.1	2318	509.7	3703	496.4	3332	482.1	2962	466.2	2592
0.6	441.6	2847	430.0	2562	417.6	2278	403.9	1993	470.7	3118	458.4	2806	445.2	2495	430.6	2183	509.0	3486	495.7	3137	481.4	2789	465.6	2440
0.7	435.6	2441	424.2	2196	412.0	1953	398.5	1708	464.3	2673	452.2	2405	439.1	2138	424.7	1871	502.1	2988	489.0	2689	474.9	2391	459.3	2092
0.8	426.2	2135	415.1	1922	403.1	1708	389.9	1495	455.3	2339	442.4	2105	429.6	1871	415.5	1637	491.2	2615	478.4	2353	464.6	2092	449.4	1830
0.9	415.5	1943	404.7	1748	393.0	1554	380.1	1360	442.9	2128	431.3	1915	418.9	1702	405.1	1490	479.0	2379	466.5	2141	453.0	1903	438.1	1665
1.0	404.7	1708	394.1	1538	382.8	1367	370.2	1196	431.3	1871	420.1	1684	408.0	1497	394.6	1310	466.5	2091	454.3	1882	441.2	1673	426.7	1474

TABLE III.—Pierced Cylinder.

($x = \frac{1}{2}$, $a = 2$, $\lambda = \frac{1}{3}$.)

Values of τ.	Charge of $\frac{1}{5}$ ($\varpi = 50$ Kilos.)								Charge of $\frac{1}{5}$ ($\varpi = 60$ Kilos.)								Charge of $\frac{1}{4}$ ($\varpi = 75$ Kilos.)							
	$\Delta = 1$		0.9		0.8		0.7		$\Delta = 1$		0.9		0.8		0.7		$\Delta = 1$		0.9		0.8		0.7	
	v	P	v	P	v	P	v	P	v	P	v	P	v	P	v	P	v	P	v	P	v	P	v	P
0.301	524.5	4254	510.7	3828	496.0	3403	480.0	2978	.....	.....	544.4	4193	528.7	3727	511.3	3261	.....	.....	.....	.....	.....	.....	.....	.....
0.400	511.2	3203	497.8	2883	483.5	2562	467.6	2242	544.9	3508	530.7	3157	515.4	2807	498.4	2456	.....	.....	.....	.....	.....	.....	539.0	2745
0.500	488.0	2562	475.2	2306	461.5	2050	446.4	1794	520.1	2807	506.5	2526	491.9	2245	475.8	1965	.....	.....	547.8	2823	532.0	2510	514.5	2196
0.600	464.0	2135	451.9	1922	438.9	1708	424.4	1495	494.5	2339	481.6	2105	467.8	1871	452.4	1637	534.8	2615	520.8	2353	505.0	2092	489.2	1830
0.700	442.1	1830	430.5	1647	418.1	1464	404.4	1311	471.2	1996	458.9	1804	445.7	1604	431.0	1404	509.6	2241	496.3	2017	482.8	1793	466.2	1569
0.800	422.0	1602	411.0	1441	399.1	1281	386.0	1121	449.8	1754	438.0	1579	425.4	1403	411.4	1228	486.4	1961	473.7	1765	460.1	1569	444.9	1373
0.900	404.4	1424	393.8	1281	382.5	1139	378.5	1000	431.0	1559	419.8	1404	407.7	1248	403.5	1091	466.1	1744	454.0	1569	440.9	1395	426.4	1220
1.000	388.3	1281	378.2	1153	367.3	1025	355.2	897	413.9	1403	403.1	1263	391.5	1123	378.6	983	447.6	1569	435.9	1412	423.3	1255	409.5	1098

We can deduce from the theoretical formulæ the following result: *Among the systems of values of Δ and τ which give the same pressure there is one for which the velocity is a maximum.*

To demonstrate it, consider formulæ (31) and (18) for velocities and pressures; putting for brevity:

$$H = A(fa)^{\frac{1}{2}}\left(\frac{\varpi}{p}\right)^{\frac{1}{4}}\left(\frac{\varpi}{c^2}\right)^{\frac{1}{10}}u^{\frac{4}{10}}, \tag{42}$$

$$h = K\frac{fa(p\varpi)^{\frac{1}{2}}}{c^2}. \tag{43}$$

These formulæ may be written under the form:

$$v = H\Delta^{\frac{1}{4}}\varphi(\tau), \tag{44}$$

$$P = \frac{h\Delta}{\tau}, \tag{45}$$

in which the variables Δ and τ are the principal quantities to consider.

Eliminating Δ between (44) and (45) we have

$$v = Hh^{-\frac{1}{4}}\tau^{-\frac{1}{4}}\varphi(\tau)P^{\frac{1}{4}},$$

whence taking account of (39) the value of $\varphi(\tau)$:

$$v = \frac{1}{3}Hh^{-\frac{1}{4}}\tau^{-\frac{5}{4}}(3\tau - \tau_1)P^{\frac{1}{4}}. \tag{46}$$

For a given value of P this value of v depends only on the variable τ and it passes through a maximum for the particular value

$$\tau_0 = \frac{5}{3}\tau_1. \tag{47}$$

The corresponding values of the density of loading and of the velocity deduced from relations (45) and (46) are:

$$\Delta_0 = \frac{5}{3}\frac{\tau_1 P}{h}, \tag{48}$$

$$v_0 = \frac{4}{3}\left(\frac{3}{5}\right)^{\frac{5}{4}}H(h\tau_1)^{-\frac{1}{4}}P^{\frac{1}{4}}. \tag{49}$$

43. Replacing τ, H, and h in (47), (48) and (49) by their values (33), (42) and (43), we have

$$\tau_0 = 5B\frac{\lambda(pu)^{\frac{1}{2}}}{c}, \tag{50}$$

$$\Delta_0 = \frac{5B}{K}\cdot\frac{\lambda c u^{\frac{1}{2}}}{fa\varpi^{\frac{1}{2}}}P, \tag{51}$$

$$v_0 = \frac{4}{3}\left(\frac{3}{5}\right)^{\frac{5}{4}}\cdot\frac{A}{(3BK)^{\frac{1}{4}}}\cdot\left(\frac{fa}{\lambda}\right)^{\frac{1}{4}}\cdot\frac{\varpi^{\frac{9}{10}}u^{\frac{11}{10}}c^{\frac{11}{10}}}{p^{\frac{1}{2}}}P^{\frac{1}{4}}, \tag{52}$$

or better, taking account of the numerical values of A, B, K (we have from (14) of Art. 10, $A = 1.569$, $B = 0.00795$, and from formula (22) of Art. 14 $K = 0.0153$),

$$\tau_0 = 0.02875 \frac{\lambda (pu)^{\frac{3}{2}}}{c}, \tag{53}$$

$$\Delta_0 = 2.598 \frac{\lambda c u^{\frac{1}{2}}}{f a \varpi^{\frac{1}{2}}} P, \tag{54}$$

$$v_0 = 7.993 \left(\frac{fa}{\lambda}\right)^{\frac{1}{4}} \cdot \frac{\varpi^{\frac{9}{10}} u^{\frac{11}{10}} c^{\frac{11}{10}}}{p^{\frac{1}{2}}} P^{\frac{1}{4}}. \tag{55}$$

44. In the greater number of cases, notably when the value of P is considerable, formula (54) gives for the density of loading values which cannot be attained in practice. The greatest value possible of v is obtained by taking Δ equal to its limiting value.

If we suppose this value to be unity, the corresponding value of τ is found from equation (45), which, making $\Delta = 1$, and replacing h by its value from (43), gives

$$\tau = K \frac{fa(p\varpi)^{\frac{1}{2}}}{c^2} \cdot \frac{1}{P}. \tag{56}$$

Under the same hypothesis, formula (8) gives for the velocity:

$$v = \frac{A}{K^{\frac{1}{2}}} \cdot \frac{\varpi^{\frac{1}{10}} u^{\frac{4}{10}} c}{p^{\frac{1}{2}}} \left[1 - \frac{B}{H} \cdot \frac{\lambda c u^{\frac{1}{2}}}{f a \varpi^{\frac{1}{2}}} P\right] P^{\frac{1}{2}}. \tag{57}$$

We may remark that in this expression the quantity between brackets is equal, according to equation (51), to $\frac{1}{2}\Delta_0$; taking this into account and replacing the coefficients K and A in (57) and (56) by their numerical values, we have the following system:

$$\tau = 0.0153 \frac{fa(p\varpi)^{\frac{1}{2}}}{c^2} \cdot \frac{1}{P}, \tag{58}$$

$$\Delta = 1, \tag{59}$$

$$v = 12.7 \frac{\varpi^{\frac{1}{10}} c^{\frac{8}{10}} u^{\frac{4}{10}}}{p^{\frac{1}{2}}} \left[1 - \frac{1}{5}\Delta_0\right] P^{\frac{1}{2}}; \tag{60}$$

which should be substituted for formulæ (53), (54) and (55) when the elements of firing are such that the value of Δ_0 (54) is greater than unity.

45. In attributing to P the limiting value which it cannot pass without compromising the safety of the piece, the above formulæ serve to calculate the greatest velocity that can be realized in a given piece, with a given form of grain, and given weights of the projectile and charge.

To this end, we calculate first the value of Δ_0 from (54).

If this value is less than 1 we adopt it for the density of loading and calculate the duration of combustion by formula (53). If, however, the deduced value of Δ is greater than 1, we take the density of loading equal to 1, and calculate the duration of combustion by formula (58).

The velocity sought is given by formula (55) in the first case and by formula (60) in the second case.

46. We give some applications o these calculations to the 30 centimeter Krupp gun.

1st. Suppose first the limit of P is fixed at 2500 kilograms per square centimeter, then with the units adopted $P = 250000$.

The values of Δ_0 for the three forms of grain already considered, and of charges of $\frac{1}{6}$, $\frac{1}{5}$, and $\frac{1}{4}$ the weight of the projectile, are summarized in the following table:

Designation of the form of the grain.	Δ_0 Values of ϖ 50 Kilos.	60 Kilos.	75 Kilos.
Cube,	1.471	1.342	1.201
Parallelopipedon,	1.379	1.259	1.126
Pierced cylinder,	0.980	0.895	0.800

The following table gives the corresponding values of τ, Δ, and the velocity v.

Designation of the form of grain.	$\varpi = 50$ Kilos.			$\varpi = 60$ Kilos.			$\varpi = 75$ Kilos.		
	τ	Δ	v	τ	Δ	v	τ	Δ	v
Cube,	1.025	1.000	426.0	1.123	1.000	449.7	1.255	1.000	477.5
Parallelopipedon,	0.683	1.000	437.6	0.749	1.000	460.4	0.837	1.000	487.4
Pierced cylinder,	0.490	0.980	485.5	0.490	0.895	505.8	0.490	0.800	531.9

2d. Suppose in the second place the limit of P reduced to 1500 kilograms per square centimeter, which corresponds very nearly to the strength of bronze pieces.

Making $P = 150000$, formula (54) gives the values of Δ_0 less than unity; which are adopted as the densities of loading, and v and τ are calculated from formulæ (55) and (53).

We thus obtain the following table:

Designation of form of grain.	$\varpi = 50$ Kilos.			$\varpi = 60$ Kilos.			$\varpi = 75$ Kilos.		
	τ	Δ	v	τ	Δ	v	τ	Δ	v
Cube,	1.469	0.882	386.1	1.469	0.805	402.3	1.469	0.720	423.0
Parallelopipedon,	0.918	0.827	392.4	0.918	0.755	408.8	0.918	0.675	429.8
Pierced cylinder,	0.490	0.588	427.3	0.490	0.537	445.1	0.490	0.480	468.1

47. From the preceding calculations we discover some interesting results.

1st. We observe that, in guns of great resistance there is no advantage, with the forms of grain used, in taking the density of loading notably less than the gravimetric density of the charge. We can reduce the density of loading and retain the same velocity by a correlative diminution of the duration of combustion, but this augments the pressure.

If we regulate the duration of combustion in a manner to retain the pressure we diminish the velocity.

In guns of little strength, there is on the other hand always an advantage in adopting densities of loading less than the gravimetric density of the charge.

2d. In bronze guns, with powder grains of the form of a cube, sphere or parallelopipedon (generally used), the velocity cannot pass beyond 400 to 410 meters with a charge of $\frac{1}{3}$ the weight of the projectile. This limit may, however, be exceeded with the same charge using pierced cylindrical or prismatic grains. The calculation indicates finally that the limit of velocities is from 420 to 430 meters, firing ordinary powder with a charge $\frac{1}{3}$ the weight of the projectile.

These various results may be generalized and extended to different calibres by the application of the principle of similitude.

48. Theory attributes a remarkable superiority, from a ballistic point of view, to pierced cylindrical or prismatic grains.

This theoretical result should, however, be accepted with some reserve. It is exclusively due to the particular law followed in the combustion of a grain, and in practice the law would not hold good in the case of the rupture or disintegration of the grains. Again, the method adopted in the packing of the grains in the cartridges influences their ignition.

For these reasons we shall only consider the figures obtained by prismatic powders as the superior limits of velocities which may be obtained.

PART V.

ADDITIONAL PRACTICAL FORMULÆ.

1. The formulæ which give the velocity and pressure in guns have already been given. To this end, a mixed method has been used; the general form of the equations having been established by theoretical considerations, and the numerical values of certain coefficients determined by experiment.

In particular, to obtain one of the coefficients, it was assumed, conformably to the results of experiments made by the committee at Gavre, that, if all the other elements are constant, the initial velocity is (1) proportional to the $\frac{6}{10}$ power of the weight of the charge, and (2) inversely proportional to the $\frac{1}{4}$ power of the capacity of the powder chamber.

2. Now, it is known that, in certain cases, it is better to take the velocity proportional to the $\frac{8}{9}$ power of the weight of the charge.*

We shall examine, in this note, the modifications which the formulæ will undergo in this hypothesis.

The form of this new discussion differs in one point from that previously adopted. It shows that, in the ordinary conditions of fire, the two empirical laws, according to which the initial velocity depends upon the charge and the volume of the powder chamber, are not distinct.

If one of them is admitted, the other results from it; this reduces the number of data which must be determined by experiment.

3. *Formula for initial velocities.*—Calling

ϖ the weight of the charge of the powder,
p the weight of the projectile,
u the length the shot moves in the gun,
c the calibre or diameter of the bore,
Δ the density of loading,
f the force of the powder,

*Memorial de l'artillerie de la marine, tome I, p. 643; tome II, p. 40; tome V, p. 401.

τ the time of combustion of a grain under atmospheric pressure,
δ the real density of the powder,
a, λ, numerical coefficients depending upon the form of the grain,
A, B, γ coefficients independent of the elements of fire;
the initial velocity may be represented by the formula (see Part IV, (6)),

$$(1) \qquad v = A\left(\frac{fa}{\tau}\right)^{\frac{1}{2}}\left(\frac{\varpi}{p}\right)^{\frac{1}{2}}\left(\frac{\varpi}{c^2}\right)^{\frac{1}{2}-\gamma}\left(\frac{1}{\Delta}-\frac{1}{\delta}\right)^{\frac{1}{2}-\gamma} u^{\gamma}\left[1 - B\,\frac{\lambda}{\tau}\,\frac{(pu)^{\frac{1}{2}}}{c}\right].$$

4. *Formula for pressures.*—The maximum mean internal pressure is given by the formula (see Part IV, (17)),

$$(2) \qquad P = K\,\frac{fa\,(p\varpi)^{\frac{1}{2}}}{\tau c^2}\left(\frac{1}{\Delta}-\frac{1}{\delta}\right)^{-\frac{1}{2}},$$

the coefficient K being, like A and B, independent of the elements of fire.

5. The expressions (1) and (2) may be simply reduced to more convenient forms, as follows: It is known that a function is sensibly constant near a maximum. Now, in the ordinary conditions of practice, the ratio $\frac{\Delta}{\delta}$ differs little from the value $\frac{1}{2}$, which gives to the function $\frac{\Delta}{\delta}\left(1-\frac{\Delta}{\delta}\right)$ its greatest value equal to $\frac{1}{4}$. We have then, nearly,

$$\frac{\Delta}{\delta}\left(1-\frac{\Delta}{\delta}\right) = \frac{1}{4},$$

and, consequently,

$$(3) \qquad \frac{1}{\Delta}-\frac{1}{\delta} = \frac{1}{4}\cdot\frac{\delta}{\Delta^2}.$$

6. *Reduction of the velocity formula.*—Recollecting the approximate relation (3), we may substitute for the factor $\left(\frac{1}{\Delta}-\frac{1}{\delta}\right)^{\frac{1}{2}-\gamma}$ of (1) the product of

$$\frac{\Delta^{2\gamma-\frac{1}{2}}}{\delta^{\gamma-\frac{1}{4}}}$$

by a numerical factor which may be compounded with A.

Also, calling s the capacity of the powder chamber, we have $\Delta = \frac{\varpi}{s}$. The formula for velocity may therefore be written,

$$(4) \qquad v = A\left(\frac{fa}{\tau}\right)^{\frac{1}{2}} \frac{\varpi^{\frac{1}{4}+\gamma} s^{\frac{1}{4}-2\gamma} u^{\gamma}}{p^{\frac{1}{4}} c^{1-2\gamma} \delta^{\gamma-\frac{1}{4}}} \left[1 - B\frac{\lambda}{\tau}\frac{(pu)^{\frac{1}{2}}}{c}\right].$$

7. It is thus apparent that the exponents of the variables ϖ and s are not independent. Putting them equal to q and r, we have

$$2q + r = 1.$$

If we suppose the velocity proportional to the $\frac{2}{3}$ power of the charge, $q = \frac{2}{3}$; and, consequently, $r = -\frac{1}{3}$. Therefore the velocity is inversely proportional to the $\frac{1}{3}$ power of the capacity of the powder chamber.

If the conditions of fire are such that we are led to ascribe to q a less value than $\frac{2}{3}$, the corresponding value of r decreases in absolute value. For example, we have $r = -\frac{1}{5}$, for $q = \frac{6}{10}$. This consequence of calculation agrees with the experimental facts.*

8. We have, $q = \frac{1}{4} + \gamma$. If then we assume $q = \frac{5}{8}$, we have $\gamma = \frac{3}{8}$. In this hypothesis, (4) becomes

$$(5) \qquad v = A\left(\frac{fa}{\tau}\right)^{\frac{1}{2}} \frac{\varpi^{\frac{5}{8}} u^{\frac{3}{8}}}{(pcs)^{\frac{1}{8}} \delta^{\frac{1}{8}}} \left[1 - B\frac{\lambda}{\tau}\frac{(pu)^{\frac{1}{2}}}{c}\right].$$

The value of the density δ varies within rather narrow limits in practice. We may then, without sensible error, neglect the corresponding variations of $\delta^{\frac{1}{8}}$, and consider this factor, reduced to its mean value, as comprised in the coefficient A.

Restoring then the density of loading, we have for the definitive formula for velocity

$$(6) \qquad v = A\left(\frac{fa}{\tau}\right)^{\frac{1}{2}} (\varpi u)^{\frac{3}{8}} \left(\frac{\Delta}{pc}\right)^{\frac{1}{8}} \left[1 - B\frac{\lambda}{\tau}\frac{(pu)^{\frac{1}{2}}}{c}\right].$$

9. *Reduction of the pressure formula.*—Replacing in (2) the factor $\frac{1}{\Delta} - \frac{1}{\delta}$ by the approximate value (3), neglecting the variations of the square root of δ, and supposing the constant factors comprised in K, we find an expression of the form

$$P = K\frac{fa\Delta\,(p\varpi)^{\frac{1}{2}}}{\tau c^2},$$

similar to formula (18) of the preceding part.

* Memorial; tome V, p. 401.

We deduce from this the two following approximate laws:

All other elements in a gun being constant, the maximum pressure is,

1st. Proportional to the $\frac{2}{3}$ power of the weight of the charge;

2d. Inversely proportional to the capacity of the powder chamber.

As this reduction of the formula for pressures results from the assumption made in No. 5, it is necessary, in order that these two laws shall be exact, that the ratio $\frac{\Delta}{\delta}$ shall not differ much from $\frac{1}{2}$.

We believe, however, that this approximation will suffice, in practical applications, in the determination of a quantity which cannot at present be precisely found.

10. *Determination of the coefficients A and B of the formula for velocity.*—The change in the formula requires that these shall be again determined. This may be done by supposing (see Part IV, Nos. 8 and 9) that the standard powder, defined by the data

$$f = 431000, \quad \tau = 0.730, \quad a = 3, \quad \lambda = 1,$$

will give, in the 24 cent. and 14 cent. guns, velocities of 437.1m. and 438.4m., in the conditions of fire specified in the table of No. 9.

We have thus two equations which give the values,

$$A = 1.691, \quad B = .00831,$$

the units being the kilogram, decimeter, and second.

11. *Formula for velocities.*—Introducing these values in (6), and calling α and β the characteristics (Part IV, No. 7) of the powder, we have

$$(7) \qquad v = 1.691\alpha\,(\varpi u)^{\frac{2}{3}}\left(\frac{\Delta}{pc}\right)^{\frac{1}{2}}\left[1 - .00831\beta\,\frac{(pu)^{\frac{1}{2}}}{c}\right].$$

Such is the new formula we wished to establish.

12. *Experimental determination of the characteristics of a powder.*—It has already been shown (Part IV, No. 17) how this may be done.

With the new formula, the process should be as follows:

Put, for shortness

$$(8) \qquad \begin{cases} 1.691\,(\varpi v)^{\frac{2}{3}}\left(\dfrac{\Delta}{pc}\right)^{\frac{1}{2}} = \dfrac{1}{X}, \\ .00831\,\dfrac{(pu)^{\frac{1}{2}}}{c} = Y, \end{cases}$$

equation (7) may then be written,

$$Xv = \alpha - \alpha\beta Y.$$

Calling X_1, Y_1; X_2, Y_2, two particular values of X and Y, and v_1 and v_2 the corresponding measured velocities, we have

$$(9) \qquad \begin{cases} X_1 v_1 = \alpha - \alpha\beta Y_1, \\ X_2 v_2 = \alpha - \alpha\beta Y_2; \end{cases}$$

which give α and β.

If we use in the determination the fire of the 24 and 14 cent. guns in the conditions already given (Part IV, No. 9), the numerical values of X and Y are as follows:

$$\text{24 cent. gun} \begin{cases} X_1 = 0.19825 \\ Y_1 = 0.25465 \end{cases}$$

$$\text{14 cent. gun} \begin{cases} X_2 = 0.24474 \\ Y_2 = 0.14144 \end{cases}$$

In particular, for the A_2B powder, giving 439.7m. in the 24 cent. and 441.6m. in the 14 cent. piece, we have the values,

$$\alpha = 1342, \quad \beta = 1.376,$$

which are the same as those found with the formula of Part IV (see Part IV, No. 19). Also, these quantities being the basis of the calculation of the characteristics of the ordinary French powders, it is evident the table of No. 22 of Part IV may be used in the practical applications of (7).

13. With regard to the figures of this table, it may be well to note the following relative to the four French powders.

The characteristics were calculated upon the assumption that the grains are parallelopipedons with square bases. We thus suppose a regularity of form which the method of screening does not always secure. In fact, the fabrication determines only one dimension, that of the thickness of the cake, with precision. Perpendicularly to the thickness, the dimensions vary irregularly within certain limits.

It may then be more exact to assume, *as a mean*, that the grain is flattened cylinder, whose radius may easily be deduced from the thickness and density, and the number of grains to the kilogram.

The values of α and β, calculated in this hypothesis, are as follows:

Name of Powder.	Characteristics. α	β
SP_2	1297	1.266
SP_1	1570	1.913
C_1	1746	2.347
C_2	2064	3.355

We shall conclude this note by deducing from (6) some relations which may be of use in practice.

14. *Maximum velocity.*—It has already been remarked (Part IV, No. 28) that, if τ is considered a variable, and all the other quantities are constants, the velocity passes through a maximum. If we call τ_1 the value of τ which corresponds to this maximum, we have (see Part IV, No. 29),

$$\tau_1 = 3B\frac{\lambda(pu)^{\frac{1}{2}}}{c},$$

and the corresponding velocity is of the form,

$$v_1 = A_1\left(\frac{fa}{\lambda}\right)^{\frac{1}{2}}\frac{\varpi^{\frac{3}{4}}\Delta^{\frac{1}{2}}c^{\frac{1}{4}}u^{\frac{1}{4}}}{p^{\frac{1}{4}}}, \tag{10}$$

the value of A_1 being*

$$A_1 = \frac{2}{3}A(3B)^{-\frac{1}{2}}.$$

This new value of v_1 shows that, if the powder used differs little from that which will give the maximum velocity, and if the travel of the shot varies between narrow limits, the *initial velocity is sensibly proportional to the $\frac{1}{4}$ power of that travel.*

15. *On the greatest value of the velocity corresponding to a given value of the maximum pressure.*—The theoretical formula for velocity leads also to the following result (Part IV, No. 42). Among the systems of values of Δ and τ which give the same pressure, there is one for which the velocity will be a maximum.

Let us call τ_0 and Δ_0 the values of τ and Δ which, for a given value of P, give the maximum velocity. We easily find,

$$\tau_0 = .04155\frac{\lambda(pu)^{\frac{1}{2}}}{c},$$

$$\Delta_0 = 2.716\frac{\lambda cu^{\frac{1}{2}}}{fa\varpi^{\frac{1}{2}}}P,$$

$$v_0 = 8.520\left(\frac{fa}{\lambda}\right)^{\frac{1}{4}}\frac{\varpi^{\frac{1}{4}}u^{\frac{1}{4}}c^{\frac{1}{2}}}{p^{\frac{1}{4}}}P^{\frac{1}{4}}.$$

16. The preceding formulæ represent the facts of experiment with an exactness which is sensibly the same as in the use of those which were established in the hypothesis that the initial velocity is proportional to the $\frac{6}{10}$ power of the charge, and inversely proportional to the $\frac{1}{4}$ power of the density of loading. They are perhaps simpler and more convenient for purposes of application.

* Replacing A and B by their values found in No. 10, $A = 1.661$, $B = 0.00831$, we find the numerical value of A_1 to be 7.140.